Journal of Computational and Applied Mathematics 313 (2017) 355–382

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Numerical evaluation of integrals involving the product of two Bessel functions and a rational fraction arising in some elastodynamic problems

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ARTICLE INFO

Article history:
Received 8 March 2016
Received in revised form 9 September 2016

Keywords:
Hankel transform
Bessel functions
Layered media
Elastodynamics

ABSTRACT

This paper presents numerical techniques for evaluating integrals of the form

$$\int_0^\infty \frac{J_\beta(k\rho)\, J_\gamma(kR)}{k^\alpha (k-s)} dk.$$

These integrals arise during the application of the Hankel transform to pass the displacements of a layered soil profile from the wave number domain to the spatial domain in three-dimensional problems of elastodynamics. The objective here is to obtain solutions with an adequate accuracy from the engineering point of view to the integrals that arise in a first order formulation of a wave propagation model widely used for layered soils.

1. Introduction

The integrals considered here arise after expressing the flexibility matrix of a stratified soil profile layer as function of modal parameters of first order in the wave number domain [1]:

$$F_{ij}(k) = \begin{bmatrix} \sum_{\mathbf{r}=1}^{N_r} \frac{\varphi^i_{\rho,\mathbf{r}}\varphi^j_{\rho,\mathbf{r}}}{k-s_\mathbf{r}} & 0 & \sum_{\mathbf{r}=1}^{N_r} \frac{\varphi^i_{\rho,\mathbf{r}}\varphi^j_{z,\mathbf{r}}}{k-s_\mathbf{r}} \\ 0 & \sum_{\boldsymbol{l}=1}^{N_l} \frac{\varphi^i_{\theta,\boldsymbol{l}}\varphi^j_{\theta,\boldsymbol{l}}}{k-s_\boldsymbol{l}} & 0 \\ \sum_{\mathbf{r}=1}^{N_r} \frac{\varphi^i_{z,\mathbf{r}}\varphi^j_{\rho,\mathbf{r}}}{k-s_\mathbf{r}} & 0 & \sum_{\mathbf{r}=1}^{N_r} \frac{\varphi^i_{z,\mathbf{r}}\varphi^j_{z,\mathbf{r}}}{k-s_\mathbf{r}} \end{bmatrix} \quad (1)$$

where k represents the wave number, s and φ are the eigenvalues and eigenvector components of different wave propagation modes, i and j represent two generic interfaces of the layered soil, ρ, θ and z are the radial, azimuthal and vertical cylindrical coordinates, respectively, while $\mathbf{r}$ and $\boldsymbol{l}$ refer to the Rayleigh and Love waves, respectively. The flexibility matrix of a layer thus expressed represents a variant of first order as an alternative to the second order formulation presented for Kausel [2]

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http://dx.doi.org/10.1016/j.cam.2016.09.043
0377-0427/

and Kausel and Roesset [3], where this matrix is also described as a combination of different propagation modes of Rayleigh wave (coupled coordinates ρ and z) and Love waves (coordinate θ). The application of uniform loads or loads with linear variation on circular areas of radius R at the interfaces of the layered profile generates the Bessel function of the 1st kind $J_\gamma(kR)$ in the expressions of displacements in the wave number domain, while the application of inverse Hankel transform to return to the spatial domain produces the Bessel function of the 1st kind $J_\beta(k\rho)$.

2. Integrals to solve

The integrals to solve have the following general form

$$V_{\alpha\beta\gamma}(\rho, R, s) = \int_0^\infty \frac{J_\beta(k\rho)J_\gamma(kR)}{k^\alpha(k-s)}dk \tag{2}$$

using a similar nomenclature to that used by Hemsley [4]. The values adopted by the parameters α, β and γ are the integer numbers 0, 1 and 2. The parameters ρ and R are nonnegative real numbers, while the eigenvalue s is in general a complex number. The integral in (2) does not have closed analytic solution, with exceptions such as for $s = 0$ that corresponds to the static solution to the elastodynamic problem, and for $R = 0$ that it is related to point loads in this type of problems.

The numerical technique proposed in this paper is explicitly used for the evaluation of the integral (2) for $\Re(s) \le 0$ (2nd and 3rd quadrants of the complex plane), where the denominator of the integrand does not vanish for any value of k, so that the integrals become non-oscillating in function of the coordinate ρ. For $\Re(s) > 0$ (1st and 4th quadrants) the solution is expressed as the sum of an integral with analytic solution and other integral with numerical solution that is obtained by changing the sign of s and using the technique for $\Re(s) \le 0$. To this end, the denominator of the integrand in (2) is replaced by

$$\frac{1}{k-s} = \frac{2s}{k^2-s^2} + \frac{1}{k+s} \tag{3}$$

for odd values of $(\alpha+\beta+\gamma)$, while this denominator is replaced by

$$\frac{1}{k-s} = \frac{2k}{k^2-s^2} - \frac{1}{k+s} \tag{4}$$

for even values of $(\alpha+\beta+\gamma)$. Thus, evaluation of the integral for $\Re(s) > 0$ is performed as

$$\begin{aligned} V_{\alpha\beta\gamma}(\rho, R, s) &= 2s\int_0^\infty \frac{J_\beta(k\rho)J_\gamma(kR)}{k^\alpha(k^2-s^2)}dk + \int_0^\infty \frac{J_\beta(k\rho)J_\gamma(kR)}{k^\alpha(k+s)}dk \\ &= 2sW_{\alpha\beta\gamma}(\rho, R, s) + V_{\alpha\beta\gamma}(\rho, R, -s) \end{aligned} \tag{5}$$

for odd values of $(\alpha+\beta+\gamma)$, or as

$$\begin{aligned} V_{\alpha\beta\gamma}(\rho, R, s) &= 2\int_0^\infty \frac{J_\beta(k\rho)J_\gamma(kR)}{k^{\alpha-1}(k^2-s^2)}dk - \int_0^\infty \frac{J_\beta(k\rho)J_\gamma(kR)}{k^\alpha(k+s)}dk \\ &= 2W_{(\alpha-1)\beta\gamma}(\rho, R, s) - V_{\alpha\beta\gamma}(\rho, R, -s) \end{aligned} \tag{6}$$

for even values of $(\alpha+\beta+\gamma)$. The application of this procedure is due to the fact that the integral

$$W_{\alpha\beta\gamma}(\rho, R, s) = \int_0^\infty \frac{J_\beta(k\rho)J_\gamma(kR)}{k^\alpha(k^2-s^2)}dk \tag{7}$$

possesses an analytic solution only for odd values of $(\alpha+\beta+\gamma)$ as in cases analyzed by Kausel [2], who presents solutions for values of s in the 4th quadrant of the complex plane. For the 1st quadrant it is required to replace the Hankel function of the 2nd kind for that of the 1st kind (as originally proposed by Watson [5]). The nature of the integrals (5) and (6) is oscillating due to the integral in (7), while the other integral of non-oscillating nature is solved by numerical techniques as proposed for $\Re(s) \le 0$.

The conditions to be satisfied by parameters α, β and γ so that the integral becomes finite are given below. The integrand in (2) for $k \to 0$ results in

$$\left.\frac{J_\beta(k\rho)J_\gamma(kR)}{k^\alpha(k-s)}\right|_{k\to 0} = \frac{\rho^\beta R^\gamma}{2^{\beta+\gamma}\beta!\,\gamma!}\frac{k^{\beta+\gamma-\alpha}}{(k-s)} \tag{8}$$

from which it follows that

$$\beta+\gamma-\alpha \ge 0. \tag{9}$$

The integrand in (2) for $k \to \infty$ yields

$$\left.\frac{J_\beta(k\rho)J_\gamma(kR)}{k^\alpha(k-s)}\right|_{k\to\infty} = \frac{2}{\pi\sqrt{\rho R}}\cos\left(k\rho - \frac{\pi}{4}(2\beta+1)\right)\cos\left(kR - \frac{\pi}{4}(2\gamma+1)\right)\frac{1}{k^{\alpha+2}} \tag{10}$$