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Immersed finite element method for eigenvalue problem

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ABSTRACT

We consider the approximation of elliptic eigenvalue problem with an interface. The main aim of this paper is to prove the stability and convergence of an immersed finite element method (IFEM) for eigenvalues using Crouzeix–Raviart P_1-nonconforming approximation. We show that spectral analysis for the classical eigenvalue problem can be easily applied to our model problem. We analyze the IFEM for elliptic eigenvalue problems with an interface and derive the optimal convergence of eigenvalues. Numerical experiments demonstrate our theoretical results.

1. Introduction

In this paper, we consider the approximation of elliptic eigenvalue problems with an interface. The interface problems are often encountered in fluid dynamics, electromagnetics, and materials science [1–4]. The main difficulty in solving such problems is caused mainly by the non-smoothness of solution across the interface. One choice to overcome it is to use finite element methods based on fitted meshes along the interface. Another choice is to use unfitted meshes independent of interface geometry for the computational domain. One of the advantages of using unfitted meshes is that we do not need to generate a mesh each time in the case of a moving interface which reduces computational costs. In this respect, several numerical methods have been proposed for example an *immersed boundary method* (IBM), *extended finite element method* (XFEM), *immersed interface method* (IIM), and *immersed finite element method* (IFEM). The IBM was introduced by Peskin to simulate cardiac mechanics and associated blood flow [5]. This method employs Eulerian and Lagrangian variables on Cartesian mesh and curvilinear mesh and they are linked by a smooth approximation of the Dirac delta function [6,7]. The XFEM is developed by extending the classical finite element method by enriching the finite element space with additional degrees of freedom [8,9]. LeVeque and Li [10] introduced the IIM based on the finite difference method where the jump conditions are properly incorporated in the scheme. However, the resulting linear system of equation from this method may not be symmetric and positive definite [11]. As an alternative, the IFEM has been developed for solving interface problems with unfitted meshes [11]. A feature of IFEM is that local basis functions are constructed to satisfy the jump conditions without additional degrees of freedom. The method has been applied to various types of partial differential equations involving interface such as two-phase incompressible flows [12] and a linear elasticity problem with a perfectly bonded interface [13,14]. The related works in this direction can be found in [15–20] and references therein.

The purpose of this paper is to prove the stability and convergence of an immersed finite element method for eigenvalues using Crouzeix–Raviart P_1-nonconforming approximation [17]. As a model problem, we consider the eigenvalue problem

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with an interface, i.e.

$$\begin{aligned} -\nabla\cdot(\beta\nabla u) &= \lambda u \quad \text{in } \Omega^+\cup\Omega^-, \\ [u]_\Gamma &= 0, \qquad \left[\beta\frac{\partial u}{\partial n}\right]_\Gamma = 0, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a convex polygonal domain in $\mathbb{R}^2$ which is separated into two subdomains Ω^+ and Ω^- by a C^2-interface $\Gamma = \partial\Omega^- \subset \Omega$ with $\Omega^+ = \Omega \setminus \Omega^-$. The symbol $[\,\cdot\,]_\Gamma$ denotes the jump across Γ. The coefficient β is bounded below and above by two positive constants,

$$0 < \beta_1 \le \beta \le \beta_2 < \infty.$$

The P_1-nonconforming FEM is widely used in solving elliptic equations and is shown to be useful in solving the mixed formulation of elliptic problems [21] and the Stokes equations [22]. Recently, Kwak et al. [17] introduced an immersed FEM based on the piecewise P_1-nonconforming polynomials and they proved optimal orders of convergence.

There have been various mathematical studies of finite element methods for eigenvalue problems. A unified approach to a posteriori and a priori error analysis for finite element approximations of self-adjoint elliptic eigenvalue problems is presented in [23]. The convergence of an adaptive method for elliptic eigenvalue problems is proved in [24]. For a nonconforming approximation, Dari et al. [25] provide a posteriori error analysis of the eigenvalue. The study of mixed eigenvalue problems can be found in [26–28]. To our best knowledge, spectral and convergence analysis of IFEM for eigenvalue problems with an interface has not been done so far. It is worth emphasizing that the spectral properties of eigenvalue problems with interface play key roles in the analysis and simulation for more complicated problems such as fluid–structure interactions, moving interfaces and the numerical stability for PDEs.

In this work, we analyze the IFEM for elliptic eigenvalue problems with interface and derive the optimal convergence of eigenvalues. Furthermore, we show that spectral analysis for the classical eigenvalue problem can be easily applied to our model problem. In particular, the spectral approximation of Galerkin methods can be proved by using fundamental properties of compact operators in Banach space. Such an investigation originates from a series of papers of Osborn and Babuška [29,30]. It has been extended in [31,32] to estimate Galerkin approximations for noncompact operators. Further application to discontinuous Galerkin approximations has been developed by Buffa et al. [33]. We formulate the eigenvalue problem with interface in terms of compact operators in order to understand the spectral behavior. The analysis presented in this paper is carried out along the lines of Refs. [31,32].

The paper is structured as follows. In the next section, we give a brief review on P_1-nonconforming IFEM [17]. In Section 3, we introduce a modified version of IFEM with an additional term and formulate the eigenvalue problem with the interface. Section 4 contains the analysis of the spectral approximation which is proved to be spurious-free and complete. The approximation is proved by means of basic results from the theory of compact operator in Banach space. In Section 5 we derive the convergence rate of eigenvalues based on P_1-nonconforming IFEM. In Section 6, we demonstrate numerical experiments for a model problem which corroborate the theoretical results in the preceding sections. In the final section, we provide a summary of our results.

2. Preliminaries

We consider an elliptic interface problem corresponding to the model problem (1):

$$\begin{aligned} -\nabla\cdot(\beta\nabla u) &= f \quad \text{in } \Omega^+\cup\Omega^-, \\ [u]_\Gamma &= 0, \qquad \left[\beta\frac{\partial u}{\partial n}\right]_\Gamma = 0, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

The weak formulation of the problem (2) is to find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \beta\nabla u\cdot\nabla v\,dx = \int_\Omega f v\,dx, \quad \forall v \in H_0^1(\Omega) \tag{3}$$

with $f \in L^2(\Omega)$.

We begin by introducing a Sobolev space which is convenient for describing the regularity of the solution of the elliptic interface problem (2). For a bounded domain D, we let $H^m(D) = W_2^m(D)$ be the usual Sobolev space of order m with semi-norm and norm denoted by $|\cdot|_{m,D}$ and $\|\cdot\|_{m,D}$, respectively. For real $m \ge 0$, the space $H^m(D)$ is defined by interpolation [34]. We define the space

$$\widetilde{H}^{1+\alpha}(D) := \{u \in H^1(D) \,:\, u \in H^{1+\alpha}(D\cap\Omega^s), s = +, -\} \quad \text{for } 0 < \alpha \le 1$$