Journal of Computational and Applied Mathematics 313 (2017) 448–453

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Uniform isochronous cubic and quartic centers: Revisited

Joan C. Artés [a], Jackson Itikawa [b], Jaume Llibre [a,*]

[a] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

[b] Departamento de Matemática, Universidade de São Paulo, 13566-590, São Carlos, São Paulo, Brazil

ARTICLE INFO

Article history:
Received 2 May 2016
Received in revised form 3 September 2016

MSC:
primary 34A36
34C07
34C25
34C29
34M35
37G15

Keywords:
Polynomial vector field
Periodic orbit
Uniform isochronous center

ABSTRACT

In this paper we completed the classification of the phase portraits in the Poincaré disc of uniform isochronous cubic and quartic centers previously studied by several authors. There are three and fourteen different topological phase portraits for the uniform isochronous cubic and quartic centers respectively.

1. Introduction and statement of the main results

The interest in the isochronous centers started in the XVII century with the works of C. Huygens, see [1]. The isochronicity phenomena appear in many physical problems, see for instance [2].

We say that $p \in \mathbb{R}^2$ is a *center* if it is a singular point of a planar differential system such that there is a neighborhood U of p where all the orbits of $U \setminus \{p\}$ are periodic. For every $q \in U \setminus \{p\}$ let $T(q)$ denote the period of the periodic orbit through q. When $T(q)$ is constant for all $q \in U \setminus \{p\}$ we say that p is an *isochronous center*. The fact that p is isochronous does not imply that the angular velocity of the vector $\overrightarrow{pq}$ is the same for all periodic orbits in $U \setminus \{p\}$. When such velocity is constant we say that p is a *uniform isochronous center* or a *rigid center*.

The uniform isochronous planar centers are characterized in the next result.

Proposition 1. *Assume that a planar polynomial differential system of degree n has a center at the origin of coordinates. Then this center is uniform isochronous if and only if by doing a linear change of variables and a scaling of time it can be written as*

$$\dot{x} = -y + x\,f(x, y), \qquad \dot{y} = x + y\,f(x, y), \tag{1}$$

with $f(x, y)$ a polynomial in x and y of degree $n - 1$, $f(0, 0) = 0$.

Proposition 1 is well-known, a proof of it can be found in [3].

The next result due to Collins [4] in 1997, also obtained by Devlin, Lloyd and Pearson [5] in 1998, and by Gasull, Prohens and Torregrosa [6] in 2005 characterizes the uniform isochronous centers of cubic polynomial systems.

* Corresponding author.
E-mail addresses: artes@mat.uab.cat (J.C. Artés), itikawa@icmc.usp.br (J. Itikawa), jllibre@mat.uab.cat (J. Llibre).

http://dx.doi.org/10.1016/j.cam.2016.09.018
0377-0427/

Theorem 2. *A cubic polynomial differential system has a uniform isochronous center at the origin if and only if it can be written as*

$$\dot{x} = -y + xf(x, y), \qquad \dot{y} = x + yf(x, y), \tag{2}$$

where $f(x, y) = a_1x + a_2y + a_3x^2 + a_4xy - a_3y^2$, *and satisfies* $a_1^2a_3 - a_2^2a_3 + a_1a_2a_4 = 0$.

We note that the proof of Theorem 2 is obtained doing an affine transformation and a rescaling of the time.

Theorem 2 can be improved as follows.

Corollary 3. *A cubic polynomial differential system has a uniform isochronous center at the origin if and only if it can be written as*

$$\dot{x} = -y + xf(x, y), \qquad \dot{y} = x + yf(x, y), \tag{3}$$

where $f(x, y) = a_1x + a_2y + a_4xy$, *and satisfies* $a_1a_2 = 0$ *and* $a_4 \neq 0$.

Corollary 3 is proved in Section 2. We note that (3) has one parameter less than in (2) and eases the polynomial condition that the parameters must hold.

Using Theorem 2 Collins obtains two normal forms for all the uniform isochronous cubic centers, one with one parameter and the other a given system. We prefer to work with the unique normal form given in Corollary 3.

In the next theorem we present the first integrals of the uniform isochronous cubic centers described by systems (3). More complicated first integrals were obtained in [7] using the normal form (2) and were given only in polar coordinates. The new normal form (3) allows to provide easier expressions of these first integrals in Cartesian coordinates.

Theorem 4. *The first integrals H of system (3) are described in what follows.*

Case 1: $\mathbf{a_1^2 + a_2^2 \neq 0}$.

Subcase 1.1: $\mathbf{4a_4 > a_1^2}$ *and* $\mathbf{a_2 = 0}$.

$$H = e^{-2\arctan\left[\frac{2a_4y+a_1}{S}\right]}\left[\frac{x^2+y^2}{a_4y^2+a_1y+1}\right]^{S/a_1},$$

where $S = \sqrt{4a_4 - a_1^2}$.

Subcase 1.2: $\mathbf{4a_4 < a_1^2}$ *and* $\mathbf{a_2 = 0}$.

$$H = \frac{(x^2+y^2)^{S/a_1}(S + a_1 + 2a_4y)^{1-S/a_1}}{(S - a_1 - 2a_4y)^{1+S/a_1}},$$

where $S = \sqrt{a_1^2 - 4a_4}$.

Subcase 1.3: $\mathbf{4a_4 < -a_2^2}$ *and* $\mathbf{a_1 = 0}$.

$$H = e^{-2\arctan\left[\frac{2a_4x+a_2}{S}\right]}\left[\frac{x^2+y^2}{a_4x^2+a_2x-1}\right]^{S/a_2},$$

where $S = \sqrt{-4a_4 - a_2^2}$.

Subcase 1.4: $\mathbf{4a_4 > -a_2^2}$ *and* $\mathbf{a_1 = 0}$.

$$H = \frac{(x^2+y^2)^{S/a_2}(S + a_2 + 2a_4x)^{1-S/a_2}}{(S - a_2 - 2a_4x)^{1+S/a_2}},$$

where $S = \sqrt{4a_4 + a_2^2}$.

Subcase 1.5: $\mathbf{4a_4 = a_1^2}$ *and* $\mathbf{a_2 = 0}$.

$$H = \frac{(x^2+y^2)e^{\frac{4}{2+a_1y}}}{(2+a_1y)^2}.$$

Subcase 1.6: $\mathbf{4a_4 = -a_2^2}$ *and* $\mathbf{a_1 = 0}$.

$$H = \frac{(x^2+y^2)e^{\frac{4}{2-a_2x}}}{(2-a_2x)^2}.$$