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On the attractivity of solutions for a class of multi-term fractional functional differential equations

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ABSTRACT

In this paper, we present some alternative results concerning with the existence and attractivity dependence of solutions for a class of nonlinear fractional functional differential equations. In our consideration, we apply the well-known Schauder fixed point theorem in conjunction with the technique of measure of noncompactness. Moreover, we provide some examples to illustrate the effectiveness of the obtained results.

1. Introduction

During the last two decades, a great interest has been devoted to the study of fractional differential equations and such class pulled the interest of so many authors towards itself, motivated by their extensive use in mathematical modelling. Fractional calculus, which is as old as classical calculus, has found important applications in the study of problems in acoustics, thermal systems, rheology or modelling of materials and mechanical systems. Moreover, in some areas of science like signal processing, identification systems, control theory or robotics, fractional differential operators seem more suitable to model than the classical integer order operators. Due to this, fractional differential equations have also been used in models about biochemistry (modelling of polymers and proteins), electrical engineering (transmission of ultrasound waves), medicine (modelling of human tissue under mechanical loads), etc. Thus, differential and integral equations of fractional order play nowadays a very important role in describing some real world phenomena.

In the recent years, the theory of fractional differential equations has been analytically investigated by a big number of very interesting and novel papers (see [1–9]). Moreover, fractional order differential equations, as functional differential equations [10], are related with causality principles and recently attractivity of solutions of fractional differential equations and fractional functional equations has been intensively investigated (see [11–17] for more details).

The aim of this paper is to study the existence of solutions of a class of multi-term fractional functional equations in the space of bounded and continuous functions on an unbounded interval. Moreover, we investigate some important properties of the solutions related with the concept of attractivity of solutions.

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Consider the initial value problem (IVP for short) of the following fractional functional differential equation:

$$\begin{cases} {}^{c}D^{\alpha}u(t) = \displaystyle\sum_{i=1}^{m} {}^{c}D^{\alpha_i} f_i(t, u_t) + f_0(t, u_t), & t > t_0, \\ u(t) = \varphi(t), & t_0 - \sigma \le t \le t_0, \end{cases} \tag{1.1}$$

where ${}^{c}D^{\alpha}$ denotes Caputo's fractional derivative of order $\alpha > 0$, σ is a positive constant, $\varphi \in C([t_0 - \sigma, t_0], \mathbb{R})$ and for each $i = 1, 2, \ldots, m$, ${}^{c}D^{\alpha_i}$ is the Caputo fractional derivative of order $0 < \alpha_i < \alpha$ and $f_i : I \times C([-\sigma, 0], \mathbb{R}) \longrightarrow \mathbb{R}$, such that $I = [t_0, \infty)$, is a given function. We also consider for any $t \in I$ the function $u_t : [-\sigma, 0] \longrightarrow \mathbb{R}$ given by $u_t(s) = u(t + s)$ for each $s \in [-\sigma, 0]$.

By using the classical Schauder fixed point principle and the concept of measure of noncompactness, we show that Eq. (1.1) has attractive solutions under rather general and convenient assumptions. We note that employing classical Schauder fixed point principle, we give an alternative result using a control function and new imposed conditions without applying the concept of measure of noncompactness.

The organization of this paper is as follows. In Section 2, we recall some useful preliminaries. In Section 3, we provide some assumptions and a lemma to present the main result of such section for Eq. (1.1) using the Schauder fixed point principle. In Section 4, we first recall some auxiliary facts about the concept of measure of noncompactness and related notations, then we study the existence of solution for Eq. (1.1) applying a generalized version of the well-known Darbo type fixed point theorem together with the technique of measure of noncompactness. Finally, in Section 5 some illustrative examples are given to show the practicability of the obtained results.

2. Preliminaries

This section is devoted to recall some essential definitions and auxiliary facts in fractional calculus. We also define some concepts related to attractivity of Eq. (1.1) together with the Schauder fixed point theorem, which will be needed further on (c.f. [6]).

Definition 2.1 (*[18,19]*). The Riemann–Liouville fractional integral of order $\gamma > 0$ with the lower limit $t_0 \in \mathbb{R}$ for a function f is defined as:

$${}^{\mathrm{RL}}I^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{1-\gamma}}\, ds, \quad t > t_0,$$

provided that the right-hand side is point-wise defined on $[t_0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 (*[18,19]*). The Riemann–Liouville fractional derivative of order $n - 1 < \gamma < n$ with the lower limit $t_0 \in \mathbb{R}$ for a function $f \in C^n([t_0, \infty), \mathbb{R})$ can be written as:

$${}^{\mathrm{RL}}D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{\gamma+1-n}}\, ds, \quad t > t_0,\ n \in \mathbb{N}.$$

Definition 2.3 (*[18]*). Caputo's fractional derivative of order $n - 1 < \gamma < n$ for a function $f \in C^n([t_0, \infty), \mathbb{R})$ can be written as

$${}^{c}D^{\gamma} f(t) = {}^{\mathrm{RL}}D^{\gamma}\left(f(s) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t_0)}{\Gamma(k-\gamma+1)} (s-t_0)^{k-\gamma} \right)(t), \quad t > t_0,\ n \in \mathbb{N}.$$

Definition 2.4 (*[16, Definition 2.5]*). The solution $u(t)$ of IVP (1.1) is attractive if there exist a constant $b_0(t_0)$ such that $|\varphi(s)| \le b_0$ (for all $s \in [t_0 - \sigma, t_0]$) implies that $u(t) \to 0$ as $t \to \infty$.

Definition 2.5 (*[17, Definition 1]*). The solution $u(t)$ of IVP (1.1) is said to be globally attractive, if there are

$$\lim_{t\to\infty} (u(t) - v(t)) = 0,$$

for any solution $v = v(t)$ of IVP (1.1).

Remark 2.6. In Section 4 we consider another definition related to concept of attractivity of solutions (see Definition 4.5).

Theorem 2.7 (*Schauder Fixed Point Theorem [20, Theorem 4.1.1]*). *Let U be a nonempty and convex subset of a normed space $\mathcal{B}$. Let T be a continuous mapping of U into a compact set $K \subset U$. Then T has a fixed point.*