Journal of Computational and Applied Mathematics 312 (2017) 13–26

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

New approach for the Fornberg–Whitham type equations

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ARTICLE INFO

Article history:
Received 18 August 2015

MSC:
47B32
46E22
74S30

Keywords:
Reproducing kernel method
Series solutions
Fornberg–Whitham equations

ABSTRACT

This paper applies the reproducing kernel Hilbert space method to the solutions of three types of Fornberg–Whitham equations: original, modified and time fractional. Comparison with Adomian decomposition method, homotopy analysis method and the variational iteration method shows the validity and applicability of the technique.

1. Introduction

The Fornberg–Whitham equation has caught much attention in recent years. Recently, there has been much work focusing on finding travelling wave solutions for this equation [1]. Eq. (1.1) was used to study the qualitative behaviour of wave-breaking [2]. Modifying the nonlinear term $u\frac{\partial u}{\partial x}$ in Eq. (1.1) to $u^2\frac{\partial u}{\partial x}$, He et al. proposed in [3] the modified Fornberg–Whitham equation. Possible travelling wave solutions of Eq. (1.1) were given in [4] and were classified in [5]. A linear dispersive Fornberg–Whitham equation was investigated in [6] where smooth and non-smooth travelling wave solutions were acquired. The global existence of solution to the viscous Fornberg–Whitham equation was shown in [7]. The boundary control of it was investigated in [8]. The variational iteration method (VIM) for the time-fractional Fornberg–Whitham equation was investigated by Saka et al. in [9].

In this paper, we consider the Fornberg–Whitham equation [10], modified Fornberg–Whitham equation [3] and time-fractional Fornberg–Whitham equation [9] of the forms:

$$\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial x^2\partial t}+\frac{\partial u}{\partial x}=u\frac{\partial^3 u}{\partial x^3}-u\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}, \tag{1.1}$$

$$\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial x^2\partial t}+\frac{\partial u}{\partial x}=u\frac{\partial^3 u}{\partial x^3}-u^2\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}, \tag{1.2}$$

and

$$\frac{\partial u^\alpha}{\partial t^\alpha}-\frac{\partial^3 u}{\partial x^2\partial t}+\frac{\partial u}{\partial x}=u\frac{\partial^3 u}{\partial x^3}-u\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}. \tag{1.3}$$

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http://dx.doi.org/10.1016/j.cam.2015.09.016
0377-0427/

The reproducing kernel Hilbert space method (RKHSM) [11] is implemented for solving these equations. Numerical experiments are shown to verify the efficiency of the RKHSM. The method is implemented for three numerical examples. The linear and nonlinear problems are solved by using RKHSM.

The theory of reproducing kernels [12] was used for the first time at the beginning of the 20th century by Zaremba. Some researchers investigate much problems by RKHSM recently [11]. For more details see [13–19].

This paper is prepared as follows. Section 2 gives useful reproducing kernel functions. The representation in $W(\Omega)$ and a related linear operator are presented in Section 3. Section 4 presents the main results. The exact and approximate solutions of Eq. (1.2) and an iterative method are developed for the kind of problems in the reproducing kernel Hilbert space. We prove that the approximate solution converges uniformly to the exact solution. Numerical experiments are shown in Section 5. Applied examples are given in Section 6. Some conclusions are given in the final section.

2. Reproducing kernel functions

We give some useful reproducing kernel functions in this space.

Definition 2.1. Let $P \neq \emptyset$. A function $Z : P \times P \to \mathbb{C}$ is called a *reproducing kernel function* of the Hilbert space H if and only if

(a) $Z(\cdot, x) \in H$ for all $x \in P$,
(b) $\langle \varrho, Z(\cdot, x)\rangle = \varrho(x)$ for all $x \in P$ and all $\varrho \in H$.

Definition 2.2. $H_2^1[0, 1]$ is defined by:

$$H_2^1[0,1] = \{v \in AC[0,1] :\ v' \in L^2[0,1]\}.$$

$$\langle v, g\rangle_{H_2^1} = v(0)g(0) + \int_0^1 v'(t)g'(t)\mathrm{d}t, \quad v, g \in H_2^1[0,1],$$

and

$$\|v\|_{H_2^1} = \sqrt{\langle v, v\rangle_{H_2^1}}, \quad v \in H_2^1[0,1],$$

are the inner product and the norm for $H_2^1[0, 1]$, respectively.

Lemma 2.3. *Reproducing kernel function q_s of reproducing kernel space $H_2^1[0, 1]$ is presented by* [11, *page* 123]*:*

$$q_s(t) = \begin{cases} 1+t, & t \le s, \\ 1+s, & t > s. \end{cases}$$

Definition 2.4. $F_2^2[0, 1]$ is given as:

$$F_2^2[0,1] = \{v \in AC[0,1] :\ v' \in AC[0,1],\ v'' \in L^2[0,1],\ v(0) = 0\}.$$

$$\langle v, g\rangle_{F_2^2} = \sum_{i=0}^{1} v^{(i)}(0)g^{(i)}(0) + \int_0^1 v''(t)g''(t)\mathrm{d}t, \quad v, g \in F_2^2[0,1],$$

and

$$\|v\|_{F_2^2} = \sqrt{\langle v, v\rangle_{F_2^2}}, \quad v \in F_2^2[0,1],$$

are the inner product and the norm of $F_2^2[0, 1]$, respectively.

Lemma 2.5. *Reproducing kernel function r_s of $F_2^2[0, 1]$ is given by* [11, *page* 148]*:*

$$r_s(t) = \begin{cases} st + \dfrac{1}{2}st^2 - \dfrac{1}{6}t^3, & t \le s, \\[2ex] ts + \dfrac{1}{2}ts^2 - \dfrac{1}{6}s^3, & t > s. \end{cases}$$

Definition 2.6. We define the space $W_2^4[0, 1]$ by

$$W_2^4[0,1] = \{v \in AC[0,1] :\ v', v'', v^{(3)} \in AC[0,1],\ v^{(4)} \in L^2[0,1]\}.$$

$$\langle v, g\rangle_{W_2^4} = \sum_{i=0}^{3} v^{(i)}(0)g^{(i)}(0) + \int_0^1 v^{(4)}(x)g^{(4)}(x)\mathrm{d}x, \quad v, g \in W_2^4[0,1],$$