Journal of Computational and Applied Mathematics 312 (2017) 39–46

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Distributed control problems for a class of degenerate semilinear evolution equations

M.V. Plekhanova *

Chelyabinsk State University, Chelyabinsk, Russia
South Ural State University, Chelyabinsk, Russia

ARTICLE INFO

Article history:
Received 30 August 2015

MSC:
49J20
35Q93
34G20
37N35

Keywords:
Control problem
Distributed control system
Semilinear degenerate evolution equation
Cauchy problem
Strong solution
Initial–boundary value problem

ABSTRACT

The unique solvability in the sense of strong solution of initial problems for a class of semilinear first order differential equations in Banach spaces with degenerate operator at the derivative is studied. It allows to prove the existence of a solution for the optimal control problem to systems, described by these initial problems. Abstract results are illustrated by the examples of degenerate systems of partial differential equations not solvable with respect to the time derivatives and of optimal control problems for them.

1. Introduction

The article presents a study of control problems for distributed systems, described by degenerate semilinear evolution equations that are not resolved with respect to the time derivative. Interest to nonlinear control problems based on their practical importance (see recent papers [1,2] and others). In Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{U}$ the statement of a problem for the operator equation

$$L\dot{x}(t) = Mx(t) + N(t, x(t)) + Bu(t), \tag{1.1}$$

$$x(t_0) = x_0, \tag{1.2}$$

$$u \in \mathcal{U}_\partial, \tag{1.3}$$

$$J(x, u) = \|x - \tilde{x}\|^2_{W_2^1(t_0,T;\mathcal{X})} + \|u - \tilde{u}\|^2_{L_2(t_0,T;\mathcal{U})}, \tag{1.4}$$

where the set of admissible controls $\mathcal{U}_\partial$ is a nonempty closed convex subset of controls space $\mathcal{U}$, functions $\tilde{x}$, $\tilde{u}$ are given, operators $B \in \mathcal{L}(\mathcal{U}; \mathcal{Y})$, $L \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, i. e. are linear and continuous. The operator L has a nontrivial kernel $\ker L \neq \{0\}$. Also, we assume that the operator M is a linear, closed, densely defined in $\mathcal{X}$ ($M \in \mathcal{C}l(\mathcal{X}; \mathcal{Y})$) and the operator of N is a nonlinear,

* Correspondence to: Laboratory of Quantum Topology, Chelyabinsk State University, Kashirin Brothers Str., 129, Chelyabinsk, Russia.
E-mail address: mariner79@mail.ru.
URL: http://www.csu.ru/Lists/List4/sotrudnik.aspx?ID=67.

http://dx.doi.org/10.1016/j.cam.2015.09.034
0377-0427/

defined on an open set $Y \subset \mathbf{R} \times \mathcal{X}$. Such problem is an abstract framework for studying of control problems for various real systems, describing by the systems of partial differential equations not solved with respect to time derivatives [3–5].

Firstly the existence of a unique strong solution of problem (1.1), (1.2) was proved in the case $\mathcal{X} = \mathcal{Y}$, $L = I$ with Caratheodory mapping N. Then these results and methods of degenerate operator semigroups theory [5] were used for investigation of problem (1.1), (1.2) solvability with degenerate operator L. Similar results were obtained in [6] but in the case of a smooth operator N. The research of initial problem (1.1), (1.2) allows to study the optimal control problem (1.1)–(1.4). Examples in the last section illustrate general results.

This work is a continuation of optimal control problems research for degenerate distributed systems in [7–9], where linear degenerate distributed control systems are considered.

2. The Cauchy problem for nondegenerate semilinear equation

Let $\mathcal{Z}$ be Banach space, $A \in \mathcal{L}(\mathcal{Z})$. Suppose that an operator $B : (t_0, T) \times \mathcal{Z} \to \mathcal{Z}$ is Caratheodory mapping, i. e. for every $z \in \mathcal{Z}$ it defines measurable mapping on (t_0, T) and for almost all $t \in (t_0, T)$ it is continuous in $z \in \mathcal{Z}$. Consider Cauchy problem

$$z(t_0) = z_0, \tag{2.1}$$

for the semilinear equation

$$\dot{z}(t) = Az(t) + B(t, z(t)). \tag{2.2}$$

A strong solution of (2.1), (2.2) on (t_0, T) is a function $z \in W_q^1(t_0, T; \mathcal{Z})$, $q \in (1, \infty)$, for which condition (2.1) and almost everywhere on (t_0, T) equality (2.2) hold.

Lemma 2.1. *Let $A \in \mathcal{L}(\mathcal{Z})$, operator $B : (t_0, T) \times \mathcal{Z} \to \mathcal{Z}$ be Caratheodory mapping, for all $z \in \mathcal{Z}$ and almost all $t \in (t_0, T)$ the estimate*

$$\|B(t, z)\|_{\mathcal{Z}} \le a(t) + c\|z\|_{\mathcal{Z}} \tag{2.3}$$

be satisfied with some $a \in L_q(t_0, T; R)$, $c > 0$. Then for every $z_0 \in \mathcal{Z}$ the function $z \in W_q^1(t_0, T; \mathcal{Z})$ is a strong solution of problem (2.1), (2.2) if and only if $z \in L_q(t_0, T; \mathcal{Z})$ and almost everywhere on (t_0, T)

$$z(t) = e^{(t-t_0)A} z_0 + \int_{t_0}^{t} e^{(t-s)A} B(s, z(s))ds. \tag{2.4}$$

Proof. Let z be a strong solution of problem (2.1), (2.2), then by condition (2.3) operator B is bounded from $L_q(t_0, T; \mathcal{Z})$ to $L_q(t_0, T; \mathcal{Z})$. Integrating equality (2.2) on the interval (t_0, t), we obtain (2.4).

Let $z \in L_q(t_0, T; \mathcal{Z})$ almost everywhere on (t_0, T) satisfy (2.4), then the function $B(\cdot, z(\cdot)) \in L_q(t_0, T; \mathcal{Z})$ and it can be checked directly that z is a strong solution of (2.1), (2.2). •

Call a mapping $B : (t_0, T) \times \mathcal{Z} \to \mathcal{Z}$ uniformly Lipschitz continuous in z, if there exists such $l > 0$, that for all (t, y), (t, z) from $(t_0, T) \times \mathcal{Z}$ the inequality $\|B(t, y) - B(t, z)\|_{\mathcal{Z}} \le l\|y - z\|_{\mathcal{Z}}$ holds. Put $\mathbf{N}_0 = \{0\} \cup \mathbf{N}$.

Theorem 2.1. *Let $A \in \mathcal{L}(\mathcal{Z})$, an operator $B : (t_0, T) \times \mathcal{Z} \to \mathcal{Z}$ be Caratheodory mapping, uniformly Lipschitz continuous in z, and $B(\cdot, v) \in L_q(t_0, T; \mathcal{Z})$ for some $v \in \mathcal{Z}$. Then for every $z_0 \in \mathcal{Z}$ problem (2.1), (2.2) has a unique strong solution on (t_0, T).*

Proof. It follows from the uniform Lipschitz continuity that $\|B(t, z)\|_{\mathcal{Z}} \le \|B(t, v)\|_{\mathcal{Z}} + l\|v\|_{\mathcal{Z}} + l\|z\|_{\mathcal{Z}}$ for all $z \in \mathcal{Z}$, a. e. on (t_0, T), therefore, condition (2.3) is performed with $a(t) = \|B(t, v)\|_{\mathcal{Z}} + l\|v\|_{\mathcal{Z}}$, $c = l$. By Lemma 2.1 it is enough to show that Eq. (2.2) has a unique solution $z \in L_q(t_0, T; \mathcal{Z})$.

In the Banach space $L_q(t_0, T; \mathcal{Z})$ we define an operator G by the equality

$$G(z)(t) = e^{(t-t_0)A} z_0 + \int_{t_0}^{t} e^{(t-s)A} B(s, z(s))ds.$$

By condition (2.3) $G : L_q(t_0, T; \mathcal{Z}) \to L_q(t_0, T; \mathcal{Z})$. As the G^r we denote the rth power of the operator G, $r \in \mathbf{N}$. If $T - t_0 < 1$ in the subsequent discussion we will replace $T - t_0$ by the unit. For almost all $t \in (t_0, T)$, $r \in \mathbf{N}$, $y, z \in L_q(t_0, T; \mathcal{Z})$ we will prove by the induction the inequality

$$\|G^r(y)(t) - G^r(z)(t)\|_{\mathcal{Z}} \le \frac{K^r (t - t_0)^{r-1/q}}{(r-1)!} \|y - z\|_{L_q(t_0,T;\mathcal{Z})}, \tag{2.5}$$

where $K = le^{(T-t_0)\|A\|_{\mathcal{L}(\mathcal{Z})}}$. While $r = 1$ with help of a Hölder's inequality we have almost everywhere on (t_0, T)

$$\|G(y)(t) - G(z)(t)\|_{\mathcal{Z}} \le le^{(T-t_0)\|A\|_{\mathcal{L}(\mathcal{Z})}} (t - t_0)^{1-1/q} \|y - z\|_{L_q(t_0,T;\mathcal{Z})}.$$