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Variational method for the solution of an inverse problem

N. Yildirim Aksoy

Department of Mathematics, Faculty of Arts and Sciences, Kafkas University, Kars, 36100, Turkey

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ABSTRACT

This paper presents a variational method for the solution of an inverse problem. An inverse problem of determining the unknown coefficient of nonlinear time-dependent Schrödinger equation is considered. Inverse problem is reformulated as a variational problem by means of an observation functional. The existence and uniqueness of solutions of the constituted variational problem are investigated. The differentiability of observation functional is obtained and a necessary condition for the solution of the variational problem is given.

1. Introduction

The inverse problems of finding the quantum-mechanical potential for Schrödinger equation which describes how the state of a physical system changes in time have drawn a lot of attention in recent years. Especially, inverse problems of determining the quantum-mechanical potential for nonlinear time dependent Schrödinger equation has a great importance in this area. It is known that the inverse problems for nonlinear time-dependent Schrödinger equation usually arise in the dispersion of light beams (waves) in nonlinear medium [1].

Let us consider the wave process describing by nonlinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t}+a_0\frac{\partial^2\psi}{\partial x^2}+ia_1(x)\frac{\partial\psi}{\partial x}-a_2(x)\psi+v(t)\psi+ia_3\left|\psi\right|^2\psi=f \tag{1.1}$$

$$\psi(x,0)=\varphi(x),\quad x\in(0,l)\,, \tag{1.2}$$

$$\psi(0,t)=\psi(l,t)=0,\quad t\in(0,T) \tag{1.3}$$

where $\psi=\psi(x,t)$ is the wave's complex amplitude, x and t are variables of space and time, respectively, $l>0,\ T>0$ are given numbers, $0\le x\le l,\ 0\le t\le T,\ i=\sqrt{-1}$ imaginary unit, $a_0, a_3>0$ are given real numbers, $a_2(x)$ is a measurable real-valued function that satisfies the condition

$$0<\mu_0\le a_2(x)\le\mu_1\quad\text{for almost all } x\in(0,l)\,,\ \mu_0,\mu_1=\text{const.}>0, \tag{1.4}$$

$a_1(x)$ is a measurable real-valued function which satisfies the condition

$$\left.\begin{array}{l}|a_1(x)|\le\mu_2,\left|\dfrac{da_1(x)}{dx}\right|\le\mu_3\quad\text{for almost all } x\in(0,l)\,,\ \mu_2,\mu_3=\text{const.}>0,\\ \qquad\qquad a_1(0)=a_1(l)=0\end{array}\right\}, \tag{1.5}$$

the functions f, φ are given complex-valued functions.

E-mail address: nyaksoy55@hotmail.com.

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When we consider the inverse problem of determining the quantum mechanical potential $v(t)$ and the function $\psi(x,t)$ from the conditions (1.1)–(1.3), the conditions (1.1)–(1.3) are not sufficient to find the solution of the inverse problem. Therefore, we use an additional condition and the choosing of an additional condition is very important to find the solution by variational methods of an inverse problem. The solutions by variational methods of the inverse problems for Schrö dinger equation were previously studied in [2–14] under different additional conditions.

As the different from the before studies, in the present paper, we consider an inverse problem for a nonlinear time-dependent Schrödinger equation in the form (1.1) from additional condition $\psi_1(x,t) = \psi_2(x,t)$ such that the functions $\psi_1(x,t) = \psi_2(x,t)$ are solutions of the following first and second type boundary value problems

$$i\frac{\partial \psi_1}{\partial t} + a_0\frac{\partial^2 \psi_1}{\partial x^2} + ia_1(x)\frac{\partial \psi_1}{\partial x} - a_2(x)\psi_1 + v(t)\psi_1 + ia_3\left|\psi_1\right|^2\psi_1 = f_1 \tag{1.6}$$

$$\psi_1(x,0) = \varphi_1(x), \quad x \in (0,l), \tag{1.7}$$

$$\psi_1(0,t) = \psi_1(l,t) = 0, \quad t \in (0,T) \tag{1.8}$$

and

$$i\frac{\partial \psi_2}{\partial t} + a_0\frac{\partial^2 \psi_2}{\partial x^2} + ia_1(x)\frac{\partial \psi_2}{\partial x} - a_2(x)\psi_2 + v(t)\psi_2 + ia_3\left|\psi_2\right|^2\psi_2 = f_2 \tag{1.9}$$

$$\psi_2(x,0) = \varphi_2(x), \quad x \in (0,l), \tag{1.10}$$

$$\frac{\partial \psi_2(0,t)}{\partial x} = \frac{\partial \psi_2(l,t)}{\partial x} = 0, \quad t \in (0,T) \tag{1.11}$$

respectively, where

$$\varphi_1 \in \mathring{W}_2^2(0,l), \qquad \varphi_2 \in W_2^2(0,l), \qquad \frac{\partial \varphi_2(0)}{\partial x} = \frac{\partial \varphi_2(l)}{\partial x} = 0, \qquad f_k \in W_2^{0,1}(\Omega) \quad \text{for } k = 1,2 \tag{1.12}$$

and $\Omega_t = (0,l)\times(0,t)$, $\Omega = \Omega_T$.

Thus, we formulate the inverse problem as follows: is to determine the unknown coefficient $v(t)$ and the functions $\psi_1(x,t)$, $\psi_2(x,t)$ from the additional condition

$$\psi_1(x,t) = \psi_2(x,t), \quad (x,t) \in \Omega. \tag{1.13}$$

The quantum mechanical potential $v(t)$ is investigated on the set

$$V \equiv \left\{ v : v(t) \in W_2^1(0,T), |v(t)| \le b_0, \left|\frac{dv(t)}{dt}\right| \le b_1 \text{ for almost all } t \in (0,T) \right\},$$

called as the set of admissible coefficients, which the constants $b_0, b_1 > 0$ are given numbers. Here the Sobolev space $W_2^1(0,T)$ is a Hilbert space consisting of all the elements $L_2(0,T)$ having square summable generalized derivatives of the first order on $(0,T)$. The Sobolev spaces $W_2^2(0,l)$, $\mathring{W}_2^2(0,l)$, $W_2^{0,1}(\Omega)$ are widely defined in [15].

Here, it should be noted that the Schrödinger equation considered in this paper contains the term $ia_1(x)\frac{\partial \psi}{\partial x}$ as different from before studies. So, this study is more comprehensive than previous.

In this paper, in Section 2, the variational formulation of the considered inverse problem and the existence of the solutions of the boundary value problems (1.6)–(1.8), (1.9)–(1.11) are given. In Section 3, the existence and uniqueness of solutions of the constituted variational problem are shown. Finally, a formula for the first variation of observation functional and a necessary condition for the solution of the variational problem are obtained in Section 4.

2. The variational formulation of the inverse problem

In this section, we will present a variational formulation of the above inverse problem. If we want to formulate the above inverse problem as a variational problem, we can write it using the method in [16] as follows: to find the minimum of the observation functional

$$J_\alpha(v) = \|\psi_1 - \psi_2\|_{L_2(\Omega)}^2 + \alpha\|v - w\|_{W_2^1(0,T)}^2 \tag{2.1}$$

on the set V under conditions (1.6)–(1.8) and (1.9)–(1.11), where $\alpha \ge 0$ is a given number, $w \in W_2^1(0,T)$ is a given element.

We must firstly prove the existence of the solutions of the boundary value problems (1.6)–(1.8) and (1.9)–(1.11) to investigate the solution of the variational problem. For this reason, let us define the solutions of the boundary value problems (1.6)–(1.8) and (1.9)–(1.11). Under given conditions, by a solution of the problem (1.6)–(1.8), we mean a function $\psi_1(x,t)$ in the space $B_1 \equiv C^0([0,T], \mathring{W}_2^2(0,l)) \cap C^1([0,T], L_2(0,l))$, which satisfies Eq. (1.6) *for almost all* $x \in (0,l)$ and *any* $t \in [0,T,]$ the initial condition (1.7) *for almost all* $x \in (0,l)$ and the boundary condition (1.8) *for almost all* $t \in (0,T)$. Similarly, by a solution of the problem (1.9)–(1.11), we mean a function $\psi_2(x,t)$ in the space $B_2 \equiv C^0([0,T], W_2^2(0,l)) \cap C^1([0,T], L_2(0,l))$,