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On a Riesz–Feller space fractional backward diffusion problem with a nonlinear source

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ABSTRACT

In this paper, a backward diffusion problem for a space-fractional diffusion equation with a nonlinear source in a strip is investigated. This problem is obtained from the classical diffusion equation by replacing the second-order space derivative with a Riesz–Feller derivative of order $\alpha \in (0, 2]$. A nonlinear problem is severely ill-posed, therefore we propose two new modified regularization solutions to solve it. We further show that the approximated problems are well-posed and their solutions converge if the original problem has a classical solution. In addition, the convergence estimates are presented under a priori bounded assumption of the exact solution. For estimating the error of the proposed method, a numerical example has been implemented.

1. Introduction

In recent decades, fractional operators have been playing more and more important roles in science and engineering [1], for instance, in the theory of viscoelasticity and viscoplasticity (mechanics), in the modeling of polymers and proteins (biochemistry), in the transmission of ultrasound waves (electrical engineering), and in the modeling of human tissue under mechanical loads (medicine), which are referred to [2–4]. These new fractional-order models are more adequate than the integer-order models, because the fractional order derivatives and integrals enable to describe the memory and hereditary properties of different substance [5].

The space-fractional diffusion equation (SFDE) has been arisen from replacing the standard space partial derivative in the diffusion equation with a space fractional partial derivative. It can be derived from the continuous-time random walk in statistical mechanics, and has a wide range of applications in the theory of probability distribution, especially in the modeling of the high-frequency price dynamics in financial markets [5,6].

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If an initial concentration distribution and boundary conditions are given, a complete recovery of the unknown solution is attainable from solving a well-posed forward problem. The well-posed forward problem, i.e. initial and boundary value problem (IBVP) for the SFDE have been studied extensively in the last few years. P. Agarwal [7–10] and his coauthors have recently developed some new methods for fractional differential equations. Liu et al. [11] considered stability and convergence of the difference methods for the space–time fractional advection–diffusion equation. Ray et al. [12] investigated application of modified decomposition method for the analytical solution of space-fractional diffusion equation. Yang et al. [13] studied several numerical methods for fractional partial differential equations with Riesz space-fractional derivatives. The space fractional diffusion also investigated by some other authors, such as M. Younis et al. [14], M. Duman et al. [15], F. Liu [16], Z. Q. Chen [17], Y. Povstenko [18]. However, in some practical problems, the boundary data on the whole boundary cannot be obtained. In situation we only may know the noisy data on a part of the boundary or at some interior points of the considered domain, which leads to an inverse problem, namely the space-fractional inverse diffusion problems (SFIDP). To the best of our knowledge, the result from the study of SFIDP is still very limited to some specific cases dealing with homogeneous problems [19–23].

Motivated by this reason, in this work, we consider a backward problem for the following nonlinear space fractional diffusion equation

$$\begin{cases} u_t(x,t) = {}_xD^{\alpha}_{\theta}u(x,t) + f(x,t,u(x,t)), & (x,t) \in \mathbb{R}\times(0,T), \\ u(x,t)|_{x\to\pm\infty} = 0, & (x,t)\in\mathbb{R}\times(0,T), \\ u(x,T) = g(x), & x\in\mathbb{R}, \end{cases} \tag{1}$$

where the space-fractional derivative ${}_xD^{\alpha}_{\theta}$ is the Riesz–Feller fractional derivative of order $\alpha (0 < \alpha \le 2)$ and skewness θ ($|\theta| \le \min\{\alpha, 2-\alpha\}, \theta \ne \pm 1$) defined in [24], as follows:

$$\begin{cases} {}_xD^{\alpha}_{\theta}f(x) = \dfrac{\Gamma(1+\alpha)}{\pi}\left\{\sin\dfrac{(\alpha+\theta)\pi}{2}\displaystyle\int_0^{\infty}\dfrac{f(x+s)-f(x)}{s^{1+\alpha}}ds \right.\\ \qquad \left. +\sin\dfrac{(\alpha-\theta)\pi}{2}\displaystyle\int_0^{\infty}\dfrac{f(x-s)-f(x)}{s^{1+\alpha}}ds\right\}, \quad 0<\alpha<2, \\ {}_xD^2_0 f(x) = \dfrac{df^2(x)}{dx^2}, \quad \alpha = 2. \end{cases}$$

The space-fractional backward diffusion problem is to determine the distribution $u(x,t)$ for $0 < t < T$ from the final concentration distribution $u(x,T)$. As it is well-known, the backward diffusion problem is severely ill-posed [21], i.e., solutions do not always exist, and in the case of existence, these do not depend continuously on the given data. In fact, from small noise contaminated physical measurements, the corresponding solutions have large errors. It makes difficult to numerical calculations. Hence, a regularization is in order.

In recent years, the homogeneous problem, i.e, $f = 0$ in Eq. (1) has been proposed by some authors. Zheng and Wei [21] used two methods, the spectral regularization and modified equation methods, to solve this problem. In [20], they developed an optimal modified method to solve this problem by an a priori and an a posteriori strategy. In 2014, Zhao et al. [23] applied a simplified Tikhonov regularization method to deal with this problem. After then, a new regularization method of iteration type for solving this problem has been introduced by Cheng et al. [19]. However, in many practical engineering applications, the diffusion occur in spatially heterogeneous environments, which requires to take account of a nonlinear source term.

As mentioned above, up-to-date we have not found any publication dealing with the SFIDP with a nonlinear source term; because it requires a special technique to deal with the fractional terms and nonlinear term for solving this nonlinear problem, this is the most challenging task. Therefore in this study, we try to develop new methods and techniques to overcome this limitation. In comparison with previous studies [19–23] on solving the space-fractional backward diffusion problem, our paper shows a significant improvement, because it can deal with the space-fractional inverse diffusion problems with nonlinear source.

This paper is organized as follows. In the following section we outline the main results. The proofs of these results is described in Section 3. In Section 4, a numerical example is proposed to show the effectiveness of the regularized methods. Then we end up the manuscript with the conclusion at Section 5.

2. The main results

Let $\widehat{g}(\omega)$ denote the Fourier transform of the integrable function $g(x)$, which defined by

$$\widehat{g}(\omega) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-ix\omega)g(x)dx, \quad i=\sqrt{-1}.$$

In terms of the Fourier transform, we have the following properties for the Riesz–Feller space-fractional derivative (see [25])

$$\widehat{{}_xD^{\alpha}_{\theta}(g)}(\omega) = -\psi^{\theta}_{\alpha}(\omega)\widehat{g}(\omega),$$