Journal of Computational and Applied Mathematics 312 (2017) 156–166

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

New solutions for first order linear fuzzy difference equations

A. Khastan

Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 45137-66731, Iran

ARTICLE INFO

Article history:
Received 16 September 2015
Received in revised form 3 March 2016

Keywords:
Fuzzy difference equations
Boundedness
Persistence
Positive equilibrium

ABSTRACT

In this paper, we study the different formulations of fuzzy difference equation $x_n = wx_{n-1} + q$, where w, q are positive fuzzy numbers. Using a generalization of division for fuzzy numbers, we investigate the existence, uniqueness and global behavior of the solution.

1. Introduction

One of the usual procedures for considering uncertainty in a dynamical system to predict the behavior of imprecise real-world phenomena is the fuzzification of the corresponding crisp difference equations and differential equations [1,2]. Independently of the similar particular formulations of the equation in any model, we expect that the solution reflects faithfully the real behavior of the system. Therefore getting different results through fuzzification of the unique crisp equation may seem unnatural and in some sense contradictory. However, we can consider this fact as an advantage of fuzzy mathematics, due to the existence of several choices which can be examined through the scrutiny of the physical features of the particular phenomena [1,3].

Fuzzy difference equation is a difference equation that initial values and parameters are fuzzy numbers and its solutions are sequences of fuzzy numbers. Recently, there has been an increasing interest in the study of fuzzy difference equations and applications of fuzzy systems [4–9]. Papaschinopoulos and Papadopoulos, in [10], studied fuzzy difference equation $x_{n+1} = A + \frac{B}{x_n}$, where x_n is a sequence of fuzzy numbers and A, B are positive fuzzy numbers. In [11], Zhang et al. studied the existence, boundedness and behavior of the positive solutions to fuzzy Riccati difference equation. Zhang et al. in [12] considered the dynamical behavior of a third order rational fuzzy difference equation using a new definition of division. Deeba and Korvin in [13] investigated the global behavior of first order linear fuzzy difference equation giving the frequency of the genotypes on the $n + 1$ generation

$$x_{n+1} = wx_n + q, \tag{1}$$

where x_n is a sequence of fuzzy numbers and w, q are positive fuzzy numbers. They applied Zadeh's Extension Principle to obtain the fuzzy solution. The solution of Eq. (1) is the sequence of fuzzy numbers satisfying in Eq. (1).

In this study, we consider two different inequivalent formulations of the crisp difference equation

$$x_n = wx_{n-1} + q, \quad q, w, x_0 \in \mathbb{R}^+, \tag{2}$$

in fuzzy setting, i.e.,

$$x_n = wx_{n-1} + q, \tag{3}$$

E-mail addresses: khastan@iasbs.ac.ir, khastan@gmail.com.

http://dx.doi.org/10.1016/j.cam.2016.03.004
0377-0427/

and

$$x_n - q = wx_{n-1}, \tag{4}$$

where x_n is a sequence of positive fuzzy numbers and $w, q, x_0 \in \mathbb{R}^+_{\mathcal{F}}$. It is easy to see that the classical solution of Eq. (2) where $w \neq 1$, is given by

$$x_n = w^n x_0 + q\frac{1-w^n}{1-w}, \quad n = 0, 1, \ldots. \tag{5}$$

The major contribution of this paper is to study the existence, uniqueness and global behavior of the solutions to Eqs. (3) and (4), using the concept of g-division [14]. Depending on the properties of the problem, we may select an appropriate formulation which reflects better the behavior of the modeled real-world problem. Similarly to the results of [3,15], we do not give an answer to the question about which is the best formulation for the problem or the closest to the classical case. In fact, we restrict our study to the analysis of the different inequivalent fuzzy versions of the crisp problem (2).

2. Preliminaries

For the convenience of readers, we give the following preliminaries, see [16,17]

Definition 2.1 (*[16]*). Consider a fuzzy subset of the real line $u : \mathbb{R} \to [0, 1]$. Then we say u is a fuzzy number if it satisfies the following properties

(i) u is normal, i.e., $\exists x_0 \in \mathbb{R}$ with $u(x_0) = 1$,
(ii) u is fuzzy convex, i.e., $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, $\forall t \in [0, 1]$, $x, y \in \mathbb{R}$,
(iii) u is upper semicontinuous on $\mathbb{R}$,
(iv) u is compactly supported i.e., $\overline{\{x \in \mathbb{R}; u(x) > 0\}}$, is compact.

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the space of all fuzzy numbers. For $0 < \alpha \leq 1$ and $u \in \mathbb{R}_{\mathcal{F}}$, we denote α-cuts of fuzzy number u by $[u]_\alpha = \{x \in \mathbb{R}; u(x) \geq \alpha\}$ and $[u]_0 = \overline{\{x \in \mathbb{R}; u(x) > 0\}}$. We call $[u]_0$, the support of fuzzy number u and denote it by $supp(u)$. For $u, v \in \mathbb{R}_{\mathcal{F}}$, $[u]_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$, $[v]_\alpha = [\underline{v}_\alpha, \overline{v}_\alpha]$ and $\lambda \in \mathbb{R}$, the sum $u + v$, the scalar product $\lambda.u$, multiplication uv and division $\frac{u}{v}$ in the standard interval arithmetic (SIA) setting are defined by

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha, \qquad [\lambda.u]_\alpha = \lambda[u]_\alpha, \quad \forall \alpha \in [0, 1],$$
$$[uv]_\alpha = [\min\{\underline{u}_\alpha\underline{v}_\alpha, \underline{u}_\alpha\overline{v}_\alpha, \overline{u}_\alpha\underline{v}_\alpha, \overline{u}_\alpha\overline{v}_\alpha\}, \max\{\underline{u}_\alpha\underline{v}_\alpha, \underline{u}_\alpha\overline{v}_\alpha, \overline{u}_\alpha\underline{v}_\alpha, \overline{u}_\alpha\overline{v}_\alpha, \}],$$
$$\left[\frac{u}{v}\right]_\alpha = \left[\min\left\{\frac{\underline{u}_\alpha}{\underline{v}_\alpha}, \frac{\underline{u}_\alpha}{\overline{v}_\alpha}, \frac{\overline{u}_\alpha}{\underline{v}_\alpha}, \frac{\overline{u}_\alpha}{\overline{v}_\alpha}\right\}, \max\left\{\frac{\underline{u}_\alpha}{\underline{v}_\alpha}, \frac{\underline{u}_\alpha}{\overline{v}_\alpha}, \frac{\overline{u}_\alpha}{\underline{v}_\alpha}, \frac{\overline{u}_\alpha}{\overline{v}_\alpha}\right\}\right], \quad 0 \notin [v]_\alpha.$$

Theorem 2.2 (*[16]*). *Let us consider the functions*

$$\underline{u}_\alpha, \overline{u}_\alpha : [0, 1] \to \mathbb{R},$$

satisfy the following conditions

(i) $\underline{u}_\alpha \in \mathbb{R}$ *is a bounded, non-decreasing, left-continuous function on* $(0, 1]$ *and it is right-continuous at* 0.
(ii) $\overline{u}_\alpha \in \mathbb{R}$ *is a bounded, non-increasing, left-continuous function on* $(0, 1]$ *and it is right-continuous at* 0.
(iii) $\underline{u}_1 \leq \overline{u}_1$.

Then there is a fuzzy number $u \in \mathbb{R}_{\mathcal{F}}$ *that has* $\underline{u}_\alpha, \overline{u}_\alpha$ *as endpoints of its* α*-cuts,* u_α.

Definition 2.3. A fuzzy number u is positive if $supp(u) \subset (0, \infty)$. We denote by $\mathbb{R}^+_{\mathcal{F}}$, the space of all positive fuzzy numbers. Similarly, we denote the space of all negative fuzzy numbers by $\mathbb{R}^-_{\mathcal{F}}$, where $u \in \mathbb{R}^-_{\mathcal{F}}$ iff $supp(u) \subset (-\infty, 0)$.

Definition 2.4 (*[17]*). Let u, v be fuzzy numbers with $[u]_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$, $[v]_\alpha = [\underline{v}_\alpha, \overline{v}_\alpha]$, $\alpha \in [0, 1]$. Then we define the metric on the fuzzy numbers space as follows

$$D(u, v) = \sup\max\{|\underline{u}_\alpha - \underline{v}_\alpha|, |\overline{u}_\alpha - \overline{v}_\alpha|\}$$

where sup is taken for all $\alpha \in [0, 1]$.

Definition 2.5. Let $x, y \in \mathbb{R}_F$. If there exists $z \in \mathbb{R}_F$ such that $x = y + z$, then z is called the H-difference of x, y and it is denoted $x \ominus y$.

We use this notation $x \ominus y$ to represent the H-difference of x and y, which is different, in general, from $x - y = x + (-1)y$.