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A new algorithm for solving dynamic equations on a time scale

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ABSTRACT

In this paper, we propose a numerical algorithm to solve a class of dynamic time scale equation which is called the q-difference equation. First, we apply the method for solving initial value problems (IVPs) which contain the first and second order delta derivatives. Illustrative examples show the usefulness of the method. Then we present applications of the method for solving the strongly non-linear damped q-difference equation. The results show that our method is more accurate than the other existing method.

1. Introduction

There have been many attempts to unify continuous and discrete mathematics. One of the most famous approaches is the time scale setting which was introduced at first by Stefan Hilger in [1] and its subsequent development in [2]. Hilger gave an efficient tool to unify continuous and discrete problems in his main theory and he extended the classical theories on an arbitrary non-empty closed subset $\mathbb{T}$ of $\mathbb{R}$, by using the standard inherited topology. Such a subset $\mathbb{T}$ is called a time scale set. Two examples of time scales sets are the set of all real numbers $\mathbb{R}$ which redounds to the definition of differential equations and the set of all integers $\mathbb{Z}$, which redounds to the definition of finite difference equations. Furthermore, there is another time scale set which is the set of q-difference equations. See, for example, [3,4]. We denote this time scale by

$$\mathbb{T} = \{q^n | n \in \mathbb{N}, 0 \le q < 1\} = \overline{q^{\mathbb{N}}}.$$

It should be noted that the time scale $\mathbb{T}$ is different from the time scale which is used in theory of quantum calculus [5]. The time scale in quantum theory is $\mathbb{R}$ rather than $\mathbb{T}$ (cf. [6]). During the past two decades, many researchers have discussed the theory and applications of this kind of time scale on dynamic equations, as well as the problem of first- and second-order linear and non-linear q-difference equations. We present a numerical method to approximate the solutions of these problems. Recent developments in the theory of approximate and numerical solutions have boosted further interest in the

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discussion of non-linear q-difference equations [5,7–11]. The most notable early work in the theory of q-difference equations are the published papers of Jackson [12,13] and Hahn [14,15], as well as more recent articles [10,16] and books [17,18] for q-special functions.

In this paper, we propose a numerical algorithm to find the numerical solution of the first- and second-order dynamic equations on $\overline{q^{\mathbb{Z}}}$ time scales which are the q-difference equations. To investigate the validity and accuracy of the method, we apply the method to find numerical approximations of two linear q-difference equations that have exact solutions. Then we solve the strongly non-linear damped q-difference equation introduced in [19] which is given by

$$y^{\Delta^2}(t) + [2\gamma + \varepsilon\gamma_1 y(t)]y^{\Delta}(t) + \Omega^2 y(t) + y^2(t) = 0, \tag{1}$$

for $t \in \overline{q^{\mathbb{N}}}$, with the initial conditions

$$y(0) = a, \qquad y^{\Delta}(0) = b,$$

where y^{Δ}, y^{Δ^2} are the first and second order delta derivatives of the unknown function y, respectively, γ and γ_1 are linear damping parameters, ε is the non-linearity parameter and Ω is the frequency of under-damped motion. The numerical results of our method compared with the results presented by Liu in [19] show the accuracy and usefulness of the method.

The construction of the paper is as follows. In Section 2, the definitions and notations of time scales are considered. In Section 3, we present a new numerical algorithm for solving the first order and second order q-difference equations. This section also contains applications to solve three illustrative examples, one of which is the strongly non-linear damped q-difference equation. Conclusions are given in Section 4.

2. Preliminaries

We first recall some basic concepts of the time scales calculus (see [7,20–22]). For a time scale $\mathbb{T}$, the forward and the backward jump operators are given by

$$\sigma : \mathbb{T} \to \mathbb{T}, \quad \sigma(t) := \inf\{z \in \mathbb{T} : z > t\}$$
$$\rho : \mathbb{T} \to \mathbb{T}, \quad \rho(t) := \sup\{z \in \mathbb{T} : z < t\}.$$

The forward step-size function $\mu : \mathbb{T} \to [0, \infty)$, is defined by

$$\mu(t) := \sigma(t) - t$$

and the backward step-size function $\nu : \mathbb{T} \to [0, \infty)$, is defined by

$$\nu(t) := t - \rho(t),$$

(see [21]). In [22] the stepsize functions are called graininess functions. Here, we follow the terminology of Ahlbrandt, Bohner and Ridenhour [23].

If $t \in \mathbb{T}$, then $\sigma(t) = t,\ \sigma(t) > t,\ \rho(t) = t,\ \rho(t) < t,\ \rho(t) < t < \sigma(t)$ and $\rho(t) = t = \sigma(t)$ are called *right dense, right scattered, left dense, left scattered, isolated* and *dense*, respectively.

A real-valued function f on $\mathbb{T}$ is called *regulated* on $\mathbb{T}$ if its right-hand limits exist at all right dense points of $T \setminus \{\sup \mathbb{T}\}$ and its left-hand limits exist at all left dense points of $T \setminus \{\inf \mathbb{T}\}$. Also f is called *right dense continuous*, or just *rd-continuous*, on $\mathbb{T}$ if it is regulated on $\mathbb{T}$ and continuous at all right dense points of $\mathbb{T}$.

For any time scale $\mathbb{T}$, the subset $\mathbb{T}^k$ is defined by

$$\mathbb{T}^k := \{t \in \mathbb{T} : t \neq \sup \mathbb{T} \text{ or } t \text{ is left dense}\}.$$

Definition 1 (*Cf. [7]*)**.** The delta derivative of $f : \mathbb{T} \to \mathbb{R}$ at $t \in \mathbb{T}^k$ is denoted by $f^{\triangle}(t)$ and it exists provided that for any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|f(\sigma(t)) - f(r) - f^{\triangle}(t)(\sigma(t) - r)| \le \varepsilon|\sigma(t) - r| \quad \text{for all } r \in U.$$

If $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$, then we say that f is Δ-differentiable on $\mathbb{T}^k$. The higher order Δ-derivatives are defined inductively as

$$y^{\Delta^0}(t) = y(t), \qquad y^{\Delta^n}(t) = (y^{\Delta^{n-1}}(t))^{\Delta}; \quad n \in \mathbb{N}.$$

Now we define the integral operator on time scales. There are many different ways to define integration on a time scale. For example, we can use the Cauchy, Riemann or Lebesgue definitions of the integrals. Here we use the concept of the Cauchy integral to define the operator of integration on time scales. For this end, first we define the Δ-antiderivative of a function on a time scale.

Definition 2. F is a Δ-antiderivative of a function $f : \mathbb{T} \to \mathbb{R}$ if $F^{\triangle}(t) = f(t)$ for $\forall\, t \in \mathbb{T}^k$.