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Exponential splitting for nonautonomous linear discrete-time systems in Banach spaces

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ABSTRACT

In this paper we consider some concepts of exponential splitting for nonautonomous linear discrete-time systems. These concepts are generalizations of some well-known concepts of (uniform and nonuniform) exponential dichotomies. Connections between these concepts are presented and some illustrating examples prove that these are distinct.

1. Introduction

The notion of exponential dichotomy introduced by O. Perron for differential equations in [1] and by Ta Li [2] for difference equations plays a central role in a large part of the theory of dynamical systems.

The notion of dichotomy for differential equations has gained prominence since the appearance of two fundamental monographs of J.L. Massera, J.J. Schäffer [3] and J.L. Daleckii, M.G. Krein [4]. These were followed by the important book of W.A. Coppel [5] who synthesized and improved the results that existed in the literature up to 1978.

The interest in the counterpart results in difference equations appeared in the paper of C.V. Coffman and J.J. Schäffer [6] and later, in 1981 when D. Henry included discrete dichotomies in his book [7]. This was followed by the classical monographs due to R.P. Agarwal [8] where the dichotomy properties of discrete-time systems are studied. Significant work was reported by C. Pötzsche in [9]. Notable contributions in dichotomy theory of discrete-time systems have been also obtained in [10–18].

The most important dichotomy concept used in the qualitative theory of dynamical systems is the uniform exponential dichotomy. In some situations, particularly in the nonautonomous setting, the concept of uniform exponential dichotomy is too restrictive and it is important to consider more general behaviors.

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Two different perspectives can be identified to generalize the concept of uniform exponential dichotomy, one can define dichotomies that depends on the initial time (and therefore are nonuniform) and, on the other hand, one can consider growth rates which do not imply an exponential dichotomy behavior, in particular exponential splitting.

The first approach leads to concepts of nonuniform exponential (respectively polynomial) dichotomies for difference equations and can be found in the works of L. Barreira, C. Valls [19,20], A. Bento, C. Silva [21,22], and L. Barreira, M. Fan, C. Valls and Z. Jimin [23].

The second approach is presented in the papers of B. Aulbach, J. Kalbrenner [24], B. Aulbach S. Siegmund [25].

In this paper we consider two concepts of exponential splitting for linear discrete-time systems in Banach spaces. These concepts use two ideas of projections sequences: invariant and strongly invariant for the respective discrete-time system, (although, in case of invertible systems, they are equivalent). These two types of projections sequences are distinct even in the finite dimensional case. For each of these concepts (exponential splitting and strong exponential splitting) we consider three important particular cases: uniform exponential splitting, exponential dichotomy and uniform exponential dichotomy respectively, uniform strong exponential splitting, strong exponential dichotomy, and uniform strong exponential dichotomy. We give characterizations of these concepts and present connections (implications and counterexamples) between them.

We note that we consider difference equations whose right-hand sides are not supposed to be invertible and the splitting concepts studied in this paper use the evolution operators in forward time. The study of noninvertible systems is of great importance and in this sense we point out the paper of B. Aulbach and J. Kalbrenner [24], where is introduced the notion of exponential forward splitting, motivated by the fact that there are differential equations whose backward solutions are not guaranteed to exist. This approach is of interest in applications, see for example, dynamical systems generated by random parabolic equations, are not invertible (for more details see [26]). Also, considering asymptotic rates of the form $e^{c\rho(n)}$, where $\rho : \mathbb{N} \to \mathbb{R}$ is an increasing function, which thus may correspond to infinite Lyapunov exponents, we obtain a concept of nonuniform exponential splitting which does not assume exponential boundedness of the splitting projections, and not only the usual exponential behavior with $\rho(n) = n$. For more details regarding the arbitrary growth rates we may refer to [27]. Also, we prove that in the particular case when the splitting projections are exponentially bounded then the two splitting concepts presented in this paper are equivalent.

2. Preliminaries

Let X be a Banach space and $B(X)$ the Banach space of all bounded linear operators on X. The norms on X and on $B(X)$ will be denoted by $\|\cdot\|$. The identity operator on X is denoted by I. If $A \in B(X)$ then we shall denote by $\operatorname{Ker} A$ the kernel of A i.e.

$$\operatorname{Ker} A = \{x \in X \text{ with } Ax = 0\}$$

respectively

$$\operatorname{Range} A = \{Ax \text{ with } x \in X\}.$$

We also denote by Δ the set of all pairs of all natural numbers (m, n) with $m \geq n$ i.e.

$$\Delta = \{(m, n) \in \mathbb{N}^2 \text{ with } m \geq n\}.$$

We also consider

$$T = \{(m, n, p) \in \mathbb{N}^3 \text{ with } m \geq n \geq p\}.$$

We consider the linear discrete-time system

$$x_{n+1} = A_n x_n, \tag{$\mathfrak{A}$}$$

where (A_n) is a sequence in $B(X)$. We associate to the system $(\mathfrak{A})$ the map

$$A_m^n = \begin{cases} A_{m-1} \cdot \ldots \cdot A_n, & \text{if } m > n \\ I, & \text{if } m = n \end{cases}$$

which is called the *evolution operator associated to* $(\mathfrak{A})$.

It is obvious that

$$A_m^n A_n^p = A_m^p, \quad \text{for all } (m, n, p) \in T$$

and every solution of $(\mathfrak{A})$ satisfies

$$x_m = A_m^n x_n \quad \text{for all } (m, n) \in \Delta.$$

If for every $n \in \mathbb{N}$ the operator A_n is invertible then the system $(\mathfrak{A})$ is called *reversible*.

Definition 1. A sequence $P : \mathbb{N} \to B(X)$ is called a *projections sequence* if

$$P(n)^2 = P(n), \quad \text{for every } n \in \mathbb{N}.$$

In what follows we denote $P(n) = P_n$ for every $n \in \mathbb{N}$.