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A generalized Lyapunov's inequality for a fractional boundary value problem[☆]

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ABSTRACT

We prove existence of positive solutions to a nonlinear fractional boundary value problem. Then, under some mild assumptions on the nonlinear term, we obtain a smart generalization of Lyapunov's inequality. The new results are illustrated through examples.

1. Introduction

Lyapunov's inequality is an outstanding result in mathematics with many different applications — see [1,2] and references therein. The result, as proved by Lyapunov in 1907 [3], asserts that if $q : [a, b] \to \mathbb{R}$ is a continuous function, then a necessary condition for the boundary value problem

$$\begin{cases} y'' + qy = 0, \quad a < t < b, \\ y(a) = y(b) = 0 \end{cases} \tag{1}$$

to have a nontrivial solution is given by

$$\int_a^b |q(s)|\, ds > \frac{4}{b-a}. \tag{2}$$

Lyapunov's inequality (2) has taken many forms, including versions in the context of fractional (noninteger order) calculus, where the second-order derivative in (1) is substituted by a fractional operator of order α.

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Theorem 1 (*See* [4]). *Consider the fractional boundary value problem*

$$\begin{cases} {}_aD^{\alpha}y + qy = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{3}$$

where ${}_aD^{\alpha}$ *is the (left) Riemann–Liouville derivative of order* $\alpha \in (1, 2]$ *and* $q : [a, b] \to \mathbb{R}$ *is a continuous function. If* (3) *has a nontrivial solution, then*

$$\int_a^b |q(s)|\, ds > \Gamma(\alpha) \left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{4}$$

A Lyapunov fractional inequality (4) can also be obtained by considering the fractional derivative in (3) in the sense of Caputo instead of Riemann–Liouville [5]. More recently, Rong and Bai obtained a Lyapunov-type inequality for a fractional differential equation but with fractional boundary conditions [6]. Motivated by [7–10] and the above results, as well as existence results on positive solutions [11–14], which are often useful in applications, we focus here on the following boundary value problem:

$$\begin{cases} {}_aD^{\alpha}y + q(t)f(y) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{5}$$

where ${}_aD^{\alpha}$ is the Riemann–Liouville derivative and $1 < \alpha \le 2$. Our first result asserts existence of nontrivial positive solutions to problem (5) (see Theorem 8). Then, under some assumptions on the nonlinear term f, we get a generalization of inequality (4) (see Theorem 10).

The paper is organized as follows. In Section 2 we recall some notations, definitions and preliminary facts, which are used throughout the work. Our results are given in Section 3: using the Guo–Krasnoselskii fixed point theorem, we establish in Section 3.1 our existence result; then, in Section 3.2, assuming that function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, concave and nondecreasing, we generalize Lyapunov's inequalities (2) and (4).

2. Preliminaries

Let $C[a, b]$ be the Banach space of all continuous real functions defined on $[a, b]$ with the norm $\|u\| = \sup_{t\in[a,b]} |u(t)|$. By $L[a, b]$ we denote the space of all real functions, defined on $[a, b]$, which are Lebesgue integrable with the norm

$$\|u\|_L = \int_a^b |u(s)|\, ds.$$

The reader interested in the fractional calculus is referred to [15]. Here we just recall the definition of (left) Riemann–Liouville fractional derivative.

Definition 2. The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $u : [a, b] \to \mathbb{R}$ is given by

$${}_aD^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{u(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$ and Γ denotes the Gamma function.

Definition 3. Let X be a real Banach space. A nonempty closed convex set $P \subset X$ is called a cone if it satisfies the following two conditions:

(i) $x \in P$, $\lambda \ge 0$, implies $\lambda x \in P$;
(ii) $x \in P$, $-x \in P$, implies $x = 0$.

Lemma 4 (*Jensen's Inequality* [16]). *Let* μ *be a positive measure and let* Ω *be a measurable set with* $\mu(\Omega) = 1$. *Let* I *be an interval and suppose that* u *is a real function in* $L(d\mu)$ *with* $u(t) \in I$ *for all* $t \in \Omega$. *If* f *is convex on* I, *then*

$$f\left(\int_{\Omega} u(t)d\mu(t)\right) \le \int_{\Omega} (f \circ u)(t)d\mu(t). \tag{6}$$

If f *is concave on* I, *then the inequality* (6) *holds with "*$\le$*" substituted by "*$\ge$*".*

Lemma 5 (*Guo–Krasnoselskii Fixed Point Theorem* [17]). *Let* X *be a Banach space and let* $K \subset X$ *be a cone. Assume* Ω_1 *and* Ω_2 *are bounded open subsets of* X *with* $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, *and let* $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ *be a completely continuous operator such that*

(i) $\|Tu\| \ge \|u\|$ *for any* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|$ *for any* $u \in K \cap \partial\Omega_2$; *or*
(ii) $\|Tu\| \le \|u\|$ *for any* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|$ *for any* $u \in K \cap \partial\Omega_2$.

Then, T *has a fixed point in* $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.