Journal of Computational and Applied Mathematics 312 (2017) 198–208

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Resonance dynamics of kinks in the sine-Gordon model with impurity, external force and damping

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ARTICLE INFO

Article history:
Received 26 September 2015
Received in revised form 10 April 2016

MSC:
35C08
35Q51
65M06

Keywords:
Modified sine-Gordon equation
Impurity
Kink
Resonance dynamics

ABSTRACT

We study the resonance dynamics of the sine-Gordon equation kinks with a point impurity. We consider the possibility of localized nonlinear waves generation on the impurities. By using analytical and numerical methods we show that the damping and external force counteract the development of kink resonant reflection from the attracting impurity. However, the underlying cause – a resonant energy exchange between solitons – still occurs.

1. Introduction

With the help of soliton dynamics one currently describes more and more physical applications in hydrodynamics, condensed matter physics, field theory, etc. (see [1–3]). The present paper deals with the modified sine-Gordon equation (MSGE)

$$u_{tt} - u_{xx} + \sin u = \Phi, \tag{1}$$

where summand Φ contains perturbations of problem. MSGE plays an important role in a number of physical problems: the domain walls in magnetics, dislocations in crystals, fluxons in Josephson junctions and crossings, etc. (see [4]). In many cases, the behavior of solitons can be described in terms of a point particle model, then their temporal evolution will be subject to the simple differential equations (see e.g. [1,2,4]). Accounting the perturbation effect leads to a significant change in the structure of solitons, which have to be described as deformable quasiparticles (see [1,2,4,5]). At the same time, the solitons internal degrees of freedom are excited, and they can play a decisive role in some physical effects. Inner modes include translational and related to long-lived oscillations of the soliton width oscillatory mode (see [1,2]).

If the study of the small perturbations influence on the MSGE solutions can be carried out by a well-developed perturbation theory for solitons [1,2,5], then the influence of large perturbations in the general case can be carried out only with the help of numerical methods (see e.g. [6–8]). Thus far quite a number of methods for numerical solving of such equations

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http://dx.doi.org/10.1016/j.cam.2016.04.013
0377-0427/

are developed. For example, in paper [9] a compact finite-difference scheme and DIRKN-method are used. The compactness of the first scheme is that its recursive formula for a new temporary layer calculation contains not more than nine template points, including the central node where the derivatives are approximated. DIRKN-method implies a class of "diagonally implicit Runge–Kutta–Nystrom methods". Paper [10] proposes to solve the MSGE by numerical method using collocation and radial basis functions. The method of lines is used in [11]. A number of studies [12–14] use spectral and pseudospectral Fourier methods for solving equations of the MSGE type. [15] presents "gridless" scheme, using the "multi-square" quasi-interpolation. This method does not require solutions of large-scale systems of linear algebraic equations. In [16,17] there used different variants of methods based on the "predictor–corrector" scheme.

The researchers interest is attracted to the question of various types of disturbances influence on the soliton dynamics (see e.g. [18–21]). Many papers are devoted to the study of the spatial modulation (heterogeneity) of the periodic potential (or presence of impurities in the system) influence on the MSGE solitons dynamics. (see e.g. [1,2,4–8,22,23]). The model of a classical particle for the kink interaction with impurity is applicable in the case when the impurity does not allow the existence of the impurity mode—localized vibrational state on the impurity (see [1,2,4]). The importance of impurity modes for kink dynamics is shown in [1,5–8,22–25].

Here we would like to note the emergence of such an interesting effect, as the reflection of the kink moving by inertia in the dissipation-free environment by an attractive impurity due to resonance energy transfer between translational kink mode and impurity mode. For real physical systems are always characterized by the presence of dissipation, which can fundamentally influence on the system behavior. In this regard, it is necessary to investigate the influence of damping and external force on the resonance effects at MSGE kinks motion of a model with an attractive impurity.

2. Main equations and analytical solution method

The MSGE that is going to be studied in this paper is given by [26–29]:

$$u_{tt} - u_{xx} + \sin u = \varepsilon\delta(x)\sin u - 2h\sin\frac{u}{2} - \alpha u_t, \tag{2}$$

where summand $\varepsilon\delta(x)$ simulates a point impurity, $\delta(x)$—Dirac delta function, ε—constant, h—parameter that determines the external force amplitude, α—damping constant. Corresponding Lagrangian and Rayleigh dissipative function for Eq. (2):

$$L = \int_{-\infty}^{+\infty}\left\{\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - [1-\varepsilon\delta(x)](1-\cos u) + 4h\cos\frac{u}{2}\right\}dx, \tag{3}$$

$$R = \int_{-\infty}^{+\infty}\frac{1}{2}\alpha u_t^2 dx. \tag{4}$$

In the case of zero right side Eq. (2) becomes the sine-Gordon equation (SGE) and has a solution in the form of a topological soliton (or kink):

$$u_0(x,t) = 4\arctan\left[\exp\left(\pm\gamma\left(x - x(t)\right)\right)\right], \tag{5}$$

where $\gamma = (1-v^2)^{-1/2}$, v—kink velocity, $x(t) = vt + x_0$—coordinate of the kink center. If the speed is low ($v \ll 1$), then $\gamma \approx 1$. Solving the linearized for small u SGE one can find an expression that describes the structure of the impurity mode:

$$u_1(x,t) = a(t)\,e^{-\varepsilon|x|/2}, \tag{6}$$

where $a(t) = a_0\cos(\Omega t + \theta_0)$, Ω—the impurity mode frequency (11), and θ_0—the initial phase.

The case of a single point impurity excluding external power and dampening has been previously studied in detail (see [1,2,5,6]). It was shown that in the case of "non-deformable kink" approach the impurity acts as a potential. Moreover, for the corresponding sign of the constant ε it acts on kink as an attractive potential, so that a soliton can be localized. For the "deformable kink" approach the effects with a resonant character occur. The possibility of impurity mode excitation on the kink scattering, resulting in a significant change of the kink dynamics results, was also taken into account. For the case of a space extended impurity the interaction of a kink with the impurity for both deformable and non-deformable kink model was also analytically and numerically studied (see [7,22–24,29–32]).

Let's consider an approximate analytical solution of the Eq. (2) by the collective variables method [1,2]. As a collective coordinates let's take the coordinate of the kink center $x(t)$ and the amplitude of the impurity mode $a(t)$. Ansatz is the sum of the kink solutions (5) and the impurity mode (6):

$$u = u_0 + u_1 = 4\arctan e^{x-x(t)} + a(t)\,e^{-\varepsilon|x|/2}. \tag{7}$$