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A feasible primal–dual interior point method for linear semidefinite programming

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ABSTRACT

In this paper, we consider a feasible primal–dual interior point method for linear semidefinite programming problem (SDP) based on Alizadeh–Haeberly–Overton (**AHO**) direction (Monteiro, 1997). Firstly, and by a new and simple technique, we establish the existence and uniqueness of optimal solution of the perturbed problem $(SDP)_\mu$ and its convergence to optimal solution of (SDP). Next, we present new different alternatives to calculate the displacement step. After, we establish the convergence of the obtained algorithm and we show that its complexity is $\mathcal{O}\left(\sqrt{n}\ln\left[\varepsilon^{-1}(\langle X^0, S^0\rangle)\right]\right)$. Finally, we present some numerical simulations which show the effectiveness of the algorithm developed in this work.

1. Introduction

Primal–dual interior point methods are originally developed to resolve linear programming problem. Their attractive theoretical and numerical properties have motivated researchers to elaborate extensions for more general classes of optimization: linear complementarity problem [1–3], convex programming [4], semidefinite programming [5–9],… .

Semidefinite programming (SDP) is an extension of linear programming where the vectors are replaced by the matrices and the non-negative orthant $\left(\mathbb{R}^n_+\right)$ by the set of symmetrical positive semidefinite matrices. (SDP) permits to solve numerous problems, as nonlinear programming problems, quadratic programming problems,… .

We distinguish two types of interior point methods for linear problems namely, projective methods and feasible or infeasible central trajectory methods.

In this paper, we are interested in solving semidefinite programming problem (SDP) by feasible central trajectory methods. We associate to this problem a perturbed problem, denoted $(SDP)_\mu$. The idea of this method consists to draw a path of the centers defined by the perturbed **KKT** optimality conditions. Then, we use Newton's method to treat the associated perturbed equations to obtain a descent direction.

We propose four different alternatives to calculate the displacement step and study the complexity analysis of the obtained algorithm. Finally, we give some numerical examples and a general conclusion to summarize the obtained results.

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We consider the following semidefinite programming problem

$$(SDP)\begin{cases}\min\langle C,X\rangle\\ \mathcal{A}X=b,\\ X\in S^n_+,\end{cases}$$

where $b\in\mathbb{R}^m$, S^n_+ denotes the cone of positive semidefinite matrices in the real space of $(n\times n)$ symmetrical matrices S^n.
$\mathcal{A}$ is a linear operator from S^n to $\mathbb{R}^m$ defined by

$$\mathcal{A}X=(\langle A_1,X\rangle,\langle A_2,X\rangle,\ldots,\langle A_m,X\rangle)^T.$$

The matrices C and A_i, $i=1,\ldots,m$, are in S^n. The scalar product of two matrices A and B in S^n is the trace of their product i.e., $\langle A,B\rangle=\mathrm{tr}(AB)=\sum_{i,j=1}^n a_{ij}b_{ij}$. The Euclidean norm of any $M\in S^n$ is $\|M\|=\max_{i=1,\ldots,n}|\lambda_i(M)|$, where $\lambda_i(M)$, $i=1,\ldots,n$, are the eigenvalues of M. The Frobenius norm of $M\in S^n$ is $\|M\|_F=\langle M,M\rangle^{\frac{1}{2}}$. For $M\in S^n$, $M\succeq 0$ $(M\succ 0)$ means M is positive semi-definite (positive definite). The set of all $(n\times n)$ matrices with real entries is denoted by $\mathbb{R}^{n\times n}$. Given $M\in\mathbb{R}^{n\times n}$, $diag(M)$ is the $(n\times n)$ diagonal matrix with diagonal entries M_{ii}. M^T denotes the transpose of $M\in\mathbb{R}^{n\times n}$.

The dual problem associated to (SDP) is defined as follows

$$(DSDP)\begin{cases}\max b^Ty\\ \mathcal{A}^*y+S=C,\\ S\in S^n_+,\end{cases}$$

where $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$ defined from $\mathbb{R}^m$ to S^n by $\mathcal{A}^*y=\sum_{i=1}^m y_iA_i$.

The sets of strictly feasible solutions of (SDP) and $(DSDP)$ are

$$\overset{0}{\mathcal{F}}(SDP)=\left\{X\in S^n_{++}:\mathcal{A}X=b\right\},$$
$$\overset{0}{\mathcal{F}}(DSDP)=\left\{(y,S)\in\mathbb{R}^m\times S^n_{++}:\mathcal{A}^*y+S=C\right\},$$

respectively, where S^n_{++} is the set of positive definite matrices of S^n.

Throughout this paper, we assume that $\overset{0}{\mathcal{F}}(SDP)\times\overset{0}{\mathcal{F}}(DSDP)$ is nonempty and that the matrices A_i, $i=1,\ldots,m$, are linearly independent. Under first assumption, it is well known that both (SDP) and $(DSDP)$ have optimal solutions $\overline{X}$ and $(\overline{S},\overline{y})$ such that $\langle C,\overline{X}\rangle=b^T\overline{y}$ i.e., the optimal values of (SDP) and $(DSDP)$ coincide. This last condition, called strong duality, can be alternatively expressed as $\langle\overline{X},\overline{S}\rangle=0$ or $\overline{X}\,\overline{S}=0$.

To study (SDP), we replace it by the perturbed equivalent problem

$$(SDP)_\mu\begin{cases}\min\left[f_\mu(X)=\langle C,X\rangle+\mu g(X)+n\mu\ln\mu\right], & \mu>0,\\ \mathcal{A}X=b,\end{cases}$$

where

$$g(X)=\begin{cases}-\ln(\det X) & \text{if } X\in S^n_{++},\\ +\infty & \text{otherwise.}\end{cases}$$

We can also study (SDP) according to its perturbed dual

$$(DSDP)_\mu\begin{cases}\max\left[g_\mu(y,S)=b^Ty-\mu g(S)-n\mu\ln\mu\right], & \mu>0,\\ \mathcal{A}^*y+S=C.\end{cases}$$

The paper is organized as follows:

We study in Section 2 the existence and uniqueness of optimal solution of the perturbed problem $(SDP)_\mu$ and we show that when $\mu\to 0$, $(SDP)_\mu$ coincides with (SDP) i.e., if $(\overline{X}_\mu,\overline{y}_\mu,\overline{S}_\mu)$ is an optimal primal–dual solution of $(SDP)_\mu$ and $(DSDP)_\mu$ respectively, then $(\overline{X},\overline{y},\overline{S})=\lim_{\mu\to 0}(\overline{X}_\mu,\overline{y}_\mu,\overline{S}_\mu)$ is an optimal primal–dual solution of (SDP) and $(DSDP)$ respectively. In Section 3, we present briefly primal–dual central trajectory method and we propose four different alternatives to compute the appropriate displacement step. In Section 4, we describe the obtained algorithm, prove its convergence, give complexity results and show that the algorithm requires at most $\mathcal{O}\left(\sqrt{n}\ln\left[\varepsilon^{-1}(\langle X^0,S^0\rangle)\right]\right)$ iterations to obtain the optimal solution. In Section 5, we present numerical tests on some different examples to illustrate the effectiveness of the four proposed approaches and we compare with the standard CVX method.

The main advantage of $(SDP)_\mu$ resides in the strict convexity of its objective function and its feasible domain. Consequently, the conditions of optimality are necessary and sufficient. This, fosters theoretical and numerical studies of the problem.

Before this, it is necessary to show that $(SDP)_\mu$ has at least an optimal solution.