Journal of Computational and Applied Mathematics 311 (2017) 54–67

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Manifold-valued subdivision schemes based on geodesic inductive averaging

Nira Dyn [a], Nir Sharon [b,*]

[a] School of Mathematical Sciences, Tel Aviv University, Tel Aviv, Israel

[b] The Program in Applied and Computational Mathematics, Princeton University, Princeton NJ, USA

ARTICLE INFO

Article history:
Received 20 August 2015
Received in revised form 8 July 2016

MSC:
40A30
65D99
41A05

Keywords:
Manifold-valued subdivision scheme
Convergence
Contractivity
Displacement-safe scheme
Inductive geodesic mean

ABSTRACT

Subdivision schemes have become an important tool for approximation of manifold-valued functions. In this paper, we describe a construction of manifold-valued subdivision schemes for geodesically complete manifolds. Our construction is based upon the adaptation of linear subdivision schemes using the notion of repeated binary averaging, where as a repeated binary average we propose to use the geodesic inductive mean. We derive conditions on the adapted schemes which guarantee convergence from any initial manifold-valued sequence. The definition and analysis of convergence are intrinsic to the manifold. The adaptation technique and the convergence analysis are demonstrated by several important examples of subdivision schemes. Two numerical examples visualizing manifold-valued curves generated by such schemes are given together with a link to the code that generated them.

1. Introduction

In recent years methods have been developed to model certain modern data as manifold data. An example of such data is the set of orientations of an aircraft, as recorded by its black box. This time series can be interpreted as data sampled from a function mapping a real interval (the time) to the Lie group of orthogonal matrices (the orientations), see e.g., [1]. Yet, classical methods for approximation cannot cope with manifold-valued functions. For instance, there is no guarantee that linear approximation methods such as polynomial or spline interpolation produce manifold values, due to the non-linearity of manifolds.

Contrary to the development of classical approximation methods and numerical analysis methods for real-valued functions, the development in the case of manifold-valued functions, which is rather recent, was mainly concerned in its first stages with advanced numerical and approximation processes, such as geometric integration of ODE on manifolds, e.g. [2], subdivision schemes on manifolds, e.g. [3–5], and wavelets-type approximation on manifolds, e.g. [1,6]. In this paper we focus on subdivision schemes.

Subdivision schemes were created originally to design geometrical models [7]. Soon, they were recognized as methods for approximation [8,9]. The important advantage of these schemes is their simplicity and locality. Namely, they are defined by repeatedly applying simple and local arithmetic averaging. This feature enables the extension of subdivision schemes to more abstract settings, such as matrices [10], sets [11], curves [12], and nets of functions [13].

* Corresponding author.
E-mail addresses: niradyn@post.tau.ac.il (N. Dyn), nsharon@math.princeton.edu (N. Sharon).

http://dx.doi.org/10.1016/j.cam.2016.07.008

For manifold valued data, [4] introduced the concept of adapting linear subdivision schemes to manifold values, in particular for Lie groups data. This paper initiated a new direction of research on manifold-valued subdivision schemes, see e.g., [14,6,5]. The adaptation of linear subdivision schemes in this paper is done by rewriting the refinement rules in repeated binary average form, and then replacing each binary average with a weighted binary geodesic average, see e.g., [10,4].

A weighted geodesic average is a generalization of the arithmetic average $(1-t)a+tb$ in Euclidean spaces, and is defined for any weight $t \in [0, 1]$ as the point on the geodesic curve between the two points to be averaged, which divides it in the ratio $\frac{t}{1-t}$ (for $t = \frac{1}{2}$ it is the midpoint). Furthermore, on several manifolds, it can also be extended to weights outside $[0, 1]$, by extrapolating the geodesic curve of two points beyond the points, see e.g., [14]. This facilitates the adaptation of interpolatory subdivision schemes which typically involve averages with negative weights. The geodesic average is also well-defined in more general spaces known as geodesic metric spaces, see e.g., [15], and our adaptation process and most of its analyses are also valid there.

The adaptation method proposed in this paper is for values from geodesically complete manifolds. It uses a specific form of repeated binary averaging—the geodesic inductive mean, which enables to deduce the contractivity of the adapted schemes obtained from the well-known interpolatory 4-point scheme [9], the 6-point Dubuc–Deslauriers scheme [8], and the first four B-spline subdivision schemes (see e.g., [16]). The contractivity is important since it is closely related to the fundamental question of convergence.

Many results in the literature of the past few years concerning the convergence and smoothness of adapted subdivision schemes, are based on proximity conditions (see [4]). A proximity condition describes a relation between the operation of an adapted subdivision scheme to the operation of its linear counterpart. Since local manifold data are nearly in a Euclidean space, the convergence results based on proximity conditions actually show that the generated values of an adapted scheme are not "too far" from those generated by its original linear scheme. Thus, these results are valid only for "dense enough data", which is, in general, a condition that is hard to quantify and depends on the properties of the underline manifold (such as its curvature).

Recently, a progress in the convergence analysis is established by several papers which address the question of convergence from all initial data. Such a result is presented in [17] for the adaptation of schemes with non-negative mask coefficients to Hadamard spaces. Results for geodesic based subdivision schemes, as well as other adaptation methods, are derived in [10] for the manifold of positive definite matrices. For the case of interpolatory subdivision schemes there are such convergence results for several different metric spaces [14,18,6]. In this paper, we present a condition, termed displacement-safe, guaranteeing that contractivity leads to convergence, for all initial data. The displacement-safe condition requires the values after one refinement to be not too far away from the values before the refinement. First we show that our adapted schemes are displacement-safe. Then, we demonstrate the analysis of contractivity on several adapted subdivision schemes, obtained from popular linear schemes, with masks of relatively small support. The contractivity guarantees the convergence of these schemes from all initial data.

The paper is organized as follows. We start in Section 2 by providing a short survey of the required background, including a summary on linear subdivision schemes, a brief review on manifolds and geodesics, and several popular approaches to the adaptation of those schemes to manifold-valued data. In Section 3 we introduce the displacement-safe condition which links contractivity and convergence. Section 4 presents our method of adaptation and the proof showing that the adapted schemes are displacement-safe. We conclude the paper in Section 5 with the adaptation of few popular schemes, prove their convergence from all initial manifold data, and demonstrate numerically the application of two different subdivision schemes to data belonging to two different manifolds.

2. Theoretical background and notation

We start by providing a few background facts together with notation on subdivision schemes, on manifolds, and on the adaptation of subdivision schemes to manifold data.

2.1. Linear univariate subdivision schemes

In the functional setting, a univariate subdivision scheme, $\mathcal{S}$, operates on a real-valued sequence $\mathbf{f} = \{f_i \in \mathbb{R} \mid i \in \mathbb{Z}\}$, by applying refinement rules that map $\mathbf{f}$ to a new sequence $\mathcal{S}(\mathbf{f})$ associated with the values in $\frac{1}{2}\mathbb{Z}$. This process is repeated infinitely and results in values defined on the dense set of dyadic real numbers. In case the values generated from any $\mathbf{f}$ by this process converge uniformly at the dyadic points to values of a continuous function, we term the subdivision scheme convergent, see e.g., [19]. A necessary and sufficient condition for the convergence of a subdivision scheme is that the sequence PL_k, $k \in \mathbb{N}$, consists of piecewise linear interpolants to each kth refined data $\{(i2^{-k}, (\mathcal{S}^k f)_i) \mid i \in \mathbb{Z}\}$, is a Cauchy sequence in the uniform norm. We denote the limit of a convergent subdivision scheme, with the refinement rules $\mathcal{S}$, generated from the initial data $\mathbf{f}$ by $\mathcal{S}^\infty(\mathbf{f})$.

A linear univariate subdivision $\mathcal{S}$ is defined by the refinement rules,

$$\mathcal{S}(\mathbf{f})_j = \sum_{i\in\mathbb{Z}} a_{j-2i} f_i, \quad j \in \mathbb{Z}, \tag{1}$$