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Approximation of reachable sets using optimal control and support vector machines

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ABSTRACT

We propose and discuss a new computational method for the numerical approximation of reachable sets for nonlinear control systems. It is based on the support vector machine algorithm and represents the set approximation as a sublevel set of a function chosen in a reproducing kernel Hilbert space. In some sense, the method can be considered as an extension to the optimal control algorithm approach recently developed by Baier, Gerdts and Xausa. The convergence of the method is illustrated numerically for selected examples.

1. Introduction

The numerical computation of reachable sets is a crucial topic in nonlinear control theory and the quantification of deterministic uncertainty in dynamical systems. Collision avoidance of manned and unmanned vehicles is one particular application that currently attracts a lot of attention (see e.g. [1] and the references therein). Standard techniques such as the set-valued Euler method [2,3] evolve a grid-based approximation of the reachable set along the relevant time interval. They are subject to the curse of dimensionality, and there is a high degree of redundancy in the computations they carry out.

Recently, a version of the set-valued Euler method was presented in [4] that tracks the boundaries of the reachable sets and uses only the boundaries of the right-hand side of the differential inclusion. With this approach, the complexity of the Euler scheme is reduced drastically in the low-dimensional setting, but only marginally in higher dimensions.

The DFOG optimal control algorithm [5], which will be discussed in more detail in Section 2.3, is another recent attempt to reduce the proportion of irrelevant computations. Every point of a grid in the relevant region of the phase space is projected to the reachable set by solving a Mayer problem. From this data, one can derive – at least theoretically – an accurate description of the reachable set. In contrast to traditional methods, there is no guarantee that the numerical optimisation routine finds a global minimum, and therefore, the algorithm is, strictly speaking, unstable. Numerical studies, however, support the usefulness of this method.

The backward reachable set of a given target set is the set of all points from which the target can be reached before (and not at) a certain time. It is therefore not exactly the kind of reachable set we consider. A modification of the viability kernel algorithm presented in [6] can be used to compute such sets. It is also natural to characterise backward reachable sets as sublevel sets of the minimal time function, which is the unique viscosity solution of a Hamilton Jacobi equation associated with the dynamics, see [7,8].

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In this paper, we propose a new approach to the calculation and representation of a reachable set approximation, motivated as an extension to the DFOG algorithm for reachable sets with nonempty interior (see Remark 4.1 for justification of this assumption). The extension consists of using the results of these optimal control problems to search for a function in a particular function space, so that the reachable set is represented as a sublevel set of this function. The function space under consideration is a reproducing kernel Hilbert space (RKHS), and the algorithm to search for this function is an adapted support vector machine (SVM) algorithm.

Our algorithm has the advantage that it is robust to a small number of errors made by the optimisation routines from the DFOG method. In addition, the function used for the reachable set approximation has a sparse representation in terms of the optimal control results, and the algorithm focuses on information provided by points that are close to the boundary of the reachable set.

2. Reachable sets and known techniques for their approximation

In the following, we give a condensed overview over basic properties of reachable sets (see Section 2.1), the currently most common numerical methods for approximating them (see Section 2.2) and the DFOG method (see Section 2.3), which is the basis of our new method.

We recall some standard definitions with regard to set representations.

Definition 2.1. Let $A, B \subset \mathbb{R}^d$ be compact sets, and $x \in \mathbb{R}^d$. The distance of a point x to the set A is defined by

$$\operatorname{dist}(x, A) := \inf_{a\in A} \|x - a\|.$$

For any $r > 0$, the r-neighbourhood of A is the set

$$B(A, r) := \{z \in \mathbb{R}^d : \operatorname{dist}(z, A) \le r\}.$$

The projection of x to A is the set of points in A that realise the infimum distance to x, i.e.

$$\operatorname{Proj}(x, A) := \{a \in A : \|x - a\| = \operatorname{dist}(x, A)\}.$$

The Hausdorff semi-distance between sets A and B is given by

$$\mathrm{d}(A, B) := \sup_{a\in A} \operatorname{dist}(a, B),$$

and the Hausdorff distance between A and B is given by

$$\mathrm{d}_H(A, B) := \max\{\mathrm{d}(A, B), \mathrm{d}(B, A)\}.$$

Throughout this paper, the symbol $\|\cdot\|$ denotes the Euclidean norm. The symbols $\|\cdot\|_\infty$, $\operatorname{dist}_\infty(x, A)$ etc. denote the corresponding concepts based on the maximum norm.

2.1. Reachable sets

Let U be a nonempty convex and compact subset of $\mathbb{R}^d$ and

$$\mathcal{U} := \left\{u \in L^\infty([t_0, T], \mathbb{R}^d) : u(t) \in U \text{ for almost all } t \in [t_0, T]\right\}$$

for fixed times $t_0 < T$. We consider the nonlinear control problem

$$\dot{x}(t) = g(t, x(t), u(t)), \quad u \in \mathcal{U}, \tag{2.1a}$$
$$x(t_0) = x_0, \tag{2.1b}$$

for some $x_0 \in \mathbb{R}^d$, where (2.1a) holds for almost every $t \in [t_0, T]$ and $x(\cdot) \in W^{1,\infty}([t_0, T], \mathbb{R}^d)$ is absolutely continuous. We are interested in the *reachable set* at time T, given by

$$\mathcal{R}(T, t_0, x_0) := \{x(T) : x(\cdot) \text{solves (2.1)}\}.$$

Problem (2.1) is equivalent to the differential inclusion

$$\dot{x}(t) \in G(t, x(t)) \tag{2.2a}$$
$$x(t_0) = x_0, \tag{2.2b}$$

with (2.2a) valid for almost all $t \in [t_0, T]$, and $G(t, x) := \bigcup_{u\in U}\{g(t, x, u)\}$.

Reachable sets of nonlinear control systems, or, equivalently, nonlinear differential inclusions, are, in general, nonconvex. It is, however, well-known, that they enjoy several favourable properties under mild assumptions imposed on the right-hand side (see e.g. [9, Corollary 7.1] and [10]):