Journal of Computational and Applied Mathematics 311 (2017) 84–99

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A modified Casteljau algorithm to solve interpolation problems on Stiefel manifolds[☆]

Krzysztof A. Krakowski [a,d], Luís Machado [c,d], Fátima Silva Leite [b,d,*], Jorge Batista [e,d]

[a] Wydział Matematyczno-Przyrodniczy, Uniwersytet Kardynała Stefana Wyszyńskiego w Warszawie, Poland
[b] Department of Mathematics, Faculty of Science and Technology, University of Coimbra, Apartado 3008, 3001-501 Coimbra, Portugal
[c] Department of Mathematics, University of Trás-os-Montes and Alto Douro (UTAD), 5001-801 Vila Real, Portugal
[d] Institute of Systems and Robotics, University of Coimbra-Pólo II, 3030-290 Coimbra, Portugal
[e] Department of Electrical Engineering and Computers, Faculty of Science and Technology, University of Coimbra, 3030-290 Coimbra, Portugal

ARTICLE INFO

Article history:
Received 23 November 2015
Received in revised form 14 July 2016

MSC:
34C40
51M35
53C22
58E10
65D18

Keywords:
Graßmann manifolds
Stiefel manifolds
Canonical metric
Quasi-geodesics
Casteljau algorithm
Interpolation problems

ABSTRACT

The main objective of this paper is to propose a new method to generate smooth interpolating curves on Stiefel manifolds. This method is obtained from a modification of the geometric Casteljau algorithm on manifolds and is based on successive quasi-geodesic interpolation. The quasi-geodesics introduced here for Stiefel manifolds have constant speed, constant covariant acceleration and constant geodesic curvature, and in some particular circumstances they are true geodesics.

1. Introduction

Stiefel and Graßmann manifolds arise naturally in several vision applications, such as machine learning and pattern recognition, since features and patterns that describe visual objects may be represented as elements in those spaces. These geometric representations facilitate the analysis of the underlying geometry of the data. The Graßmann manifold is the space of k-dimensional subspaces in $\mathbb{R}^n$ and the Stiefel manifold is the space of k orthonormal vectors in $\mathbb{R}^n$. While a point in the Graßmann manifold represents a subspace, a point in the Stiefel manifold identifies exactly what frame (basis of vectors) is used to specify that subspace.

☆ This work was developed under FCT project PTDC/EEA-CRO/122812/2010.
* Corresponding author at: Department of Mathematics, Faculty of Science and Technology, University of Coimbra, Apartado 3008, 3001-501 Coimbra, Portugal.

E-mail addresses: k.krakowski@uksw.edu.pl (K.A. Krakowski), lmiguel@utad.pt (L. Machado), fleite@mat.uc.pt (F. Silva Leite), batista@isr.uc.pt (J. Batista).

http://dx.doi.org/10.1016/j.cam.2016.07.018
0377-0427/

Although these two manifolds are related, the geometry of the Graßmann manifold is much simpler than that of the Stiefel manifold. This reflects on solutions of simple formulated problems, such as the case of geodesics that join two given points. A formula for the geodesic that joins two points on Graßmann manifolds and depends explicitly only on those points was recently presented in [1]. Knowing such explicit formulas is also a crucial step to solve other important problems such as, averaging, fitting and interpolation of data. Results about geodesics on Stiefel manifolds are not so easy to obtain. Even the simpler problem of finding a geodesic that starts at a given point with a prescribed velocity is not so straightforward, as can be seen for instance in the work of Edelman et al. [2]. In the present paper we solve a slightly different but related problem, which consists of joining two points on the Stiefel manifold by quasi-geodesics. These curves have constant speed, constant covariant acceleration and, therefore, constant geodesic curvature. Moreover, they are defined explicitly in terms of the points they join. In some cases, depending on those points, the quasi-geodesics are true geodesics. Interestingly enough, these special curves can be used successfully to generate smooth interpolating curves on the Stiefel manifold, as will be explained later. These results may have a great impact in computer vision, since a curve that interpolates a set of time-labelled points on the Stiefel manifold may correspond, for instance, to the temporal evolution of an event or dynamic scene from which only a limited number of observations was captured, as nicely explained in [3].

The organisation of this paper is the following. After this introduction that motivates the reader to the importance of the problems studied here in the context of applications, we introduce in Section 2 the manifolds that play a major role throughout the paper: Graßmann and Stiefel manifolds. This section also includes known results about geodesics and, in particular, a closed formula for the geodesic in the Graßmann manifold that joins two given points. In Section 3 we present quasi-geodesics in the Stiefel manifold. These curves have some interesting properties, such as constant speed, constant covariant acceleration and constant geodesic curvature. We provide an explicit formula for quasi-geodesics that join two arbitrary points in the Stiefel manifold and show that in two particular circumstances they are true geodesics. Interpolation problems are formulated and solved in the next three sections of the paper. First, we review in Section 4 the Casteljau algorithm on manifolds and then implement this algorithm to generate a $\mathcal{C}^1$-smooth interpolating curve on the Graßmann manifold satisfying some prescribed boundary conditions. This algorithm, first introduced in [4] and later explored for instance in [5], [6], and [7], is a generalisation to manifolds of the classical algorithm to generate Bézier curves in $\mathbb{R}^n$, which was derived independently by De Casteljau [8] and Bézier [9]. The algorithm is based on successive geodesic interpolation, so its implementation requires that explicit formulas for the geodesic joining two points are known. Due to the work in [1], the implementation of the Casteljau algorithm is possible on Graßmann manifolds, but not on Stiefel manifolds. To overcome this difficulty, we explain in Section 5 how one can modify the Casteljau algorithm in a way that will prove to be very important in Section 6, where an interpolation problem on the Stiefel manifold is solved intrinsically, that is, without resorting to other spaces. This is done using a convenient modification of the Casteljau algorithm, which consists of replacing successive geodesic interpolation by successive quasi-geodesic interpolation. This overcomes the difficulties that arise from not knowing explicit formulas for geodesics that join two arbitrary points on the Stiefel manifold and justifies the introduction of quasi-geodesics. In Section 7 we include the results of two experiments to illustrate the loss of smoothness when instead of the interpolating method proposed in Section 6 for data on the Stiefel manifold one uses interpolating methods performed on the Graßmann manifold. The paper ends with some concluding remarks.

2. Preliminaries

In this section we recall the main definitions associated to Graßmann and Stiefel manifolds and several properties that will be used throughout this paper. Due to the important role that these manifolds play in applied areas, they have been studied in the context of numerical algorithms, for instance in [2], [10] and [11], and in a more abstract form in [12]. Recently, Batzies et al. [1] found a closed form expression for a geodesic in the Graßmann manifold that joins two given points. This formula turns out to be very important for the developments throughout the whole paper. If not stated otherwise, the definitions and concepts in this section are taken from Refs. [2] and [1].

2.1. The Graßmann manifold

Let $\mathfrak{s}(n)$ and $\mathfrak{so}(n)$ denote the set of all $n \times n$ real symmetric matrices and the set of all $n \times n$ real skew-symmetric matrices respectively.

The (real) Graßmann manifold $\mathcal{G}_{n,k}$ is the set of all k-dimensional linear subspaces in $\mathbb{R}^n$, where $n \geq k \geq 1$. This manifold has a matrix representation

$$\mathcal{G}_{n,k} := \left\{ P \in \mathfrak{s}(n) : \; P^2 = P \text{ and } \operatorname{rank}(P) = k \right\}$$

so that it is considered a submanifold of $\mathbb{R}^{n \times n}$ with dimension $k(n-k)$. Graßmann manifold $\mathcal{G}_{n,k}$ can also be viewed as a homogeneous space

$$\mathcal{G}_{n,k} \cong \mathbb{O}(n)/(\mathbb{O}(k) \times \mathbb{O}(n-k)),$$

where $\mathbb{O}(n)$ is the orthogonal Lie group.