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Solving steady incompressible Navier–Stokes equations by the Arrow–Hurwicz method

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ABSTRACT

This article is devoted to analyzing an Arrow–Hurwicz type method for solving incompressible Navier–Stokes equations discretized by mixed element methods. Under several reasonable conditions, it is proved by a subtle argument that the method converges geometrically with a contraction number independent of the finite element mesh size h, even for regular triangulations. A series of numerical examples are provided to illustrate the computational performance of the method.

1. Introduction

Numerical solution of steady incompressible Navier–Stokes equations play fundamental roles in computational fluid dynamics and engineering applications (cf. [1–3]). The mathematical model describing the steady flow of an incompressible Newtonian fluid (such as air or water) is given as follows (cf. [4,3]):

$$\begin{cases} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2 \text{ or } 3)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\nu = 1/Re > 0$ indicates the viscosity coefficient (Re: the Reynolds number), and $\boldsymbol{f}$ is the prescribed body force; $\boldsymbol{u}$ and p are the corresponding velocity field and pressure field, respectively. To simplify the discussion, we impose the non-slip condition for Eqs. (1.1),

$$\boldsymbol{u} = \boldsymbol{0}. \tag{1.2}$$

Since the pressure p is unique up to a constant, we assume

$$p \in L_0^2(\Omega) := \left\{ q \in L^2(\Omega);\ \int_\Omega q dx = 0 \right\}. \tag{1.3}$$

In order to derive the variational form of the Navier–Stokes equations (1.1) satisfying the boundary condition (1.2) and the constraint (1.3), let us first introduce some notation about Sobolev spaces. Given a non-negative integer m, let $H^m(\Omega)$ be the usual Sobolev space consisting of all functions $v \in L^2(\Omega)$ whose weak derivatives with the total degree no more than m are still $L^2(\Omega)$-integrable. We equip $H^m(\Omega)$ with the standard norm $\|\cdot\|_m$ and seminorm $|\cdot|_m$ (cf. [5]). The closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_m$ is denoted by $H_0^m(\Omega)$. The dual of $H_0^m(\Omega)$ is denoted by $H^{-m}(\Omega)$. Let $\boldsymbol{H}^m(\Omega)$ be the product

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space $(H^m(\Omega))^d$, whose induced norm, seminorm, and scalar product are expressed with the same symbols over $H^m(\Omega)$, when there is no confusion caused. The similar conventions also apply to $H_0^m(\Omega)$ and $H^{-m}(\Omega)$.

Next, write $\boldsymbol{V} := \boldsymbol{H}_0^1(\Omega)$, $P := L_0^2(\Omega)$, and let $\boldsymbol{V}^3$ be the product space $\boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V}$. For any $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ in $\boldsymbol{V}$, define

$$a_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}dx,$$
$$N(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) = \frac{1}{2}a_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) - \frac{1}{2}a_1(\boldsymbol{u};\boldsymbol{w},\boldsymbol{v}).$$

By integration by parts, it is easy to check that the above two trilinear forms are identical over $\boldsymbol{V}^3$, so $N(\cdots)$ may be viewed as the anti-symmetrization of $a_1(\cdot;\cdot,\cdot)$. Thus, the variational form of problem (1.1)–(1.3) reads as follows (cf. [6,2,3]).

Problem Q. Find $(\boldsymbol{u}, p) \in \boldsymbol{V} \times P$ such that

$$\begin{cases} N(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) - (p, \operatorname{div}\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall \boldsymbol{v}\in\boldsymbol{V}, & \text{(a)}\\ (\operatorname{div}\boldsymbol{u}, q) = 0 \quad \forall p \in P, & \text{(b)}\end{cases} \tag{1.4}$$

where and in what follows, $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, $(\cdot,\cdot)$ denotes the usual scalar product over $L^2(\Omega)$ and $\langle\cdot,\cdot\rangle$ the bilinear form between the dual pair $\boldsymbol{H}^{-1}(\Omega)$ and $\boldsymbol{H}_0^1(\Omega)$.

As shown in [6,2,3], there exists a positive number $\mathcal{N}$ such that

$$|a_1(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})| \le \mathcal{N}|\boldsymbol{u}|_1|\boldsymbol{v}|_1|\boldsymbol{w}|_1 \quad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in \boldsymbol{H}_0^1(\Omega). \tag{1.5}$$

Furthermore, define

$$\Lambda = \nu^{-2}\mathcal{N}\|\boldsymbol{f}\|_{-1}, \tag{1.6}$$

where

$$\|\boldsymbol{f}\|_{-1} := \sup_{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)} \frac{\langle \boldsymbol{f},\boldsymbol{v}\rangle}{|\boldsymbol{v}|_1}.$$

Then, as shown in [6,2,3], problem Q has a solution for any $\boldsymbol{f}\in\boldsymbol{H}^{-1}(\Omega)$, and the solution is unique whenever $\Lambda < 1$.

Based on the variational form (1.4)(a)–(1.4)(b), we are able to develop mixed element methods for solving problem (1.1)–(1.3). Let $\mathcal{T}_h = \{K\}_{K\in\mathcal{T}_h}$ be a regular family of triangulations of Ω; h denotes the mesh size of $\mathcal{T}_h$ (cf. [7,8]). With each triangulation $\mathcal{T}_h$, we associate a pair of finite element spaces $(\boldsymbol{V}_h, P_h)$ such that $\boldsymbol{V}_h \subset \boldsymbol{V}$ and $P_h \subset P$. We call the pair $(\boldsymbol{V}_h, P_h)$ is stable whenever there exists a generic constant $\beta > 0$, independent of h, such that the following inf–sup condition holds:

$$\inf_{q\in P_h}\sup_{\boldsymbol{v}\in\boldsymbol{V}_h}\frac{(\operatorname{div}\boldsymbol{v}, q)}{|\boldsymbol{v}|_1\|q\|_0} \ge \beta. \tag{1.7}$$

The typical stable pairs of $(\boldsymbol{V}_h, P_h)$ include the MINI element, Girault–Raviart element, and $P_k - P_{k-1}$ element (cf. [9,6]). The $P_2 - P_1$ element is also called the Taylor–Hood element. Thus, the mixed element method for (1.4)(a)–(1.4)(b) is given as follows.

Problem Q_h. Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times P_h$ such that

$$\begin{cases} N(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}) + \nu(\nabla\boldsymbol{u}_h,\nabla\boldsymbol{v}) - (p_h, \operatorname{div}\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall \boldsymbol{v}\in\boldsymbol{V}_h, & \text{(a)}\\ (\operatorname{div}\boldsymbol{u}_h, q) = 0 \quad \forall p \in P_h. & \text{(b)}\end{cases} \tag{1.8}$$

In addition to the inf–sup condition (1.7), if $\Lambda < 1$ and the pair $(\boldsymbol{V}_h, P_h)$ satisfy the usual approximation property of finite element spaces (cf. [7,8]), then we know that problem Q_h has a unique solution and the corresponding error estimates are available (cf. [3]). Throughout this paper we will always assume that problem Q_h has a unique solution, to simplify the discussion.

Due to the importance of the problem Q_h, it is a very hot topic to develop related efficient numerical solvers (cf. [1]). As far as we know, one typical and well-used iterative method for the previous problem requires to solve the Oseen equations (or equivalently, the nonsymmetric saddle-point systems) at each iteration step. Then the saddle-point systems are solved by the preconditioned GMRES method combined with some efficient preconditioners. We refer the reader to [1,10] for an excellent survey along this line. Some more recent methods can also be found in [11,12].

On the other hand, in the past decade, He and his research group have developed another type of numerical methods for solving problem Q_h. Three iterative methods were proposed in [13] for solving problem Q_h in two-dimensional case, where some discrete Stokes equations, discrete linearized Navier–Stokes equations or discrete Oseen equations must be solved at each iteration step. More recently, several two-level iterative methods were designed in [14] for solving the previous problem in two and three dimensional cases, by combining different methods in [13] in fine and coarse meshes technically for different values of Λ given by (1.6).

In this article, our study follows a different point of view. We intend to use a novel iterative method (cf. [3]) to solve problem Q_h and analyze its convergence rate. Historically, Temam (cf. [3]) mentioned the method, called the Arrow–Hurwicz