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A new contractive condition approach to φ-fixed point results in metric spaces and its applications

Pathaithep Kumrod, Wutiphol Sintunavarat *

Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathumthani 12121, Thailand

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ABSTRACT

In this work, we introduce the new concept of a generalization of (F, φ)-contraction mapping and establish some existence results of φ-fixed point for such mappings. We also state some illustrative examples to support our results and give numerical experiment for approximating the φ-fixed point in this example. As an application, the obtained results are used to deduce some fixed point theorems in partial metric spaces and the application to nonlinear integral equations is given in order to illustrate the effectiveness of the main results.

1. Introduction and preliminaries

1.1. Partial metric spaces

In 1994, the notion of a partial metric space was introduced by Matthews [1]. He also extended the well-known Banach fixed point theorem from metric spaces to partial metric spaces. We start by recalling some definitions and properties of partial metric space.

Definition 1.1 (*[1]*). Let X be a nonempty set. A mapping $p : X \times X \to [0, \infty)$ is said to be a partial metric (briefly, p-metric) if and only if for any $x, y, z \in X$ the following conditions hold:

(p_1) $p(x, x) = p(y, y) = p(x, y) \iff x = y$ (equality);
(p_2) $p(x, x) \leq p(x, y)$ (small self-distances);
(p_3) $p(x, y) = p(y, x)$ (symmetry);
(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ (triangularity).

The pair (X, p) is also called a partial metric space.

Each partial metric p on a nonempty set X generates a T_0-topology τ_p on X which has the family of open p-balls $\{B_p(x, \varepsilon) : x \in X : \varepsilon > 0\}$, where

$$B_p(x, \varepsilon) := \{y \in X : p(x, y) < p(x, x) + \varepsilon\},$$

for all $x \in X$ and $\xi > 0$, forms a base of τ_p (see in [1]).

* Corresponding author.
E-mail addresses: phai.pathaithep@gmail.com (P. Kumrod), wutiphol@mathstat.sci.tu.ac.th, poom_teun@hotmail.com (W. Sintunavarat).

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Definition 1.2 (*[1]*). Let (X, p) be a partial metric space.

1. A sequence $\{x_n\} \subseteq X$ is called convergent to a point $x \in X$ with respect to p if and only if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$.
2. A sequence $\{x_n\} \subseteq X$ is said to be Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
3. The partial metric space (X, p) is said to be complete if and only if every Cauchy sequence $\{x_n\}$ in X converges to some $x \in X$ such that $\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x)$.

Note that a metric is evidently a p-metric. However, a p-metric on X need not be a metric on X. A classic example of partial metric space is the pair $([0, \infty), p)$, where $p(x, y) := \max\{x, y\}$ for all $x, y \in [0, \infty)$. It is easy to see that $p(x, y)$ may not be 0 whenever $x = y$. Then it is not a metric on X. For further examples, we refer to [1].

Remark 1.3 (*[1]*). If p is a partial metric on a nonempty set X, then the function $d_p : X \times X \to [0, \infty)$ defined by

$$d_p(x, y) := 2p(x, y) - p(x, x) - p(y, y) \quad \text{for all } x, y \in X \tag{1.1}$$

is a metric on X.

Lemma 1.4 (*[1]*). *Let (X, p) be a partial metric space. Then the following assertions hold:*

(i) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_p);*
(ii) *the partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete;*
(iii) *for each sequence $\{x_n\}$ in X and $x \in X$,*

$$\lim_{n\to\infty} d_p(x_n, x) = 0 \Longleftrightarrow p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

1.2. φ-fixed points and (F, φ) contraction mappings

Let X be a nonempty set, $\varphi :\to [0, \infty)$ be a given function and $T : X \to X$ be a mapping. We denote the set of all fixed points of T by

$$F_T := \{x \in X : Tx = x\}$$

and denote the set of all zeros of the function φ by

$$Z_\varphi := \{x \in X : \varphi(x) = 0\}.$$

In 2014, Jleli et al. [2] introduced the concepts of φ-fixed points, φ-Picard mappings and weakly φ-Picard mappings as follows:

Definition 1.5 (*[2]*). Let X be a nonempty set and $\varphi : X \to [0, \infty)$ be a given function. An element $z \in X$ is called φ-fixed point of the mapping $T : X \to X$ if and only if z is a fixed point of T and $\varphi(z) = 0$.

Definition 1.6 (*[2]*). Let (X, d) be a metric space and $\varphi : X \to [0, \infty)$ be a given function. A mapping $T : X \to X$ is said to be a φ-Picard mapping if and only if

(i) $F_T \cap Z_\varphi = \{z\}$, where $z \in X$,
(ii) $T^n x \to z$ as $n \to \infty$, for each $x \in X$.

Definition 1.7 (*[2]*). Let (X, d) be a metric space and $\varphi : X \to [0, \infty)$ be a given function. We say that the mapping $T : X \to X$ is a weakly φ-Picard mapping if and only if

(i) T has at least one φ-fixed point,
(ii) the sequence $\{T^n x\}$ converges for each $x \in X$, and the limit is a φ-fixed point of T.

Also, Jleli et al. introduced the new concept of control function $F : [0, \infty)^3 \to [0, \infty)$ satisfying the following conditions:

(F1) $\max\{a, b\} \leq F(a, b, c)$ for all $a, b, c \in [0, \infty)$;
(F2) $F(0, 0, 0) = 0$;
(F3) F is continuous.

Throughout this paper, unless otherwise specified, the class of all functions satisfying the conditions (F1)–(F3) is denoted by $\mathcal{F}$.

Example 1.8 (*[2]*). Let $F_1, F_2, F_3 : [0, \infty)^3 \to [0, \infty)$ be defined by

$$F_1(a, b, c) = a + b + c,$$
$$F_2(a, b, c) = \max\{a, b\} + c,$$
$$F_3(a, b, c) = a + a^2 + b + c$$

for all $a, b, c \in [0, \infty)$. Then $F_1, F_2, F_3 \in \mathcal{F}$.