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Operational Tau method for singular system of Volterra integro-differential equations

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ABSTRACT

The Legendre spectral Tau matrix formulation is proposed to approximate solution of singular system of Volterra integro-differential equations. The existence and uniqueness solution of this system are investigated by means of the υ-smoothing property of a Volterra integral operator and some projectors. The L^2—convergence of the numerical method is analyzed. It is proved theoretically and demonstrated numerically that the proposed method converges exponentially. Finally, two numerical examples illustrate the theoretical results.

1. Introduction

System of integro-differential equations arise in many mathematical modeling processes such as population growth, one dimensional viscoelasticity and reactor dynamics [1–4]. Singular systems of Volterra integro-differential equations or Volterra integro-differential–algebraic equations (**IDAEs**) are encountered as a differential–algebraic system together with an integral operator which arise in modeling nonlinear electric chains with after-effect [5,6]. In this paper, we consider numerical method for solving the singular systems of Volterra integro-differential equations in the following form:

$$L[X(t)] = A(t)X'(t) + B(t)X(t) + \int_0^t K(t,s)X(s)ds = f(t), \quad X(0) = X_0, \tag{1.1}$$

with $t \in \Omega = [0, 1]$ and $f : \Omega \to R^d (d \geq 1)$. $A(t)$ and $B(t)$ are given $d \times d$ matrices. $K(t, s)$ is the kernel matrix defined in the domain $D = \{(t, s) : 0 \leq s \leq t \leq 1\}$ and $X : \Omega \to R^d$ is the unknown function. We assume that $Rank(A) \geq 1$ and

$$detA(t) = 0, \quad \forall t \in \Omega.$$

The semi-explicit form of the system (1.1) can be described by

$$A(t) = dig(I_{d_1}, O_{d_2}), \quad d_1 + d_2 = d.$$

An initial investigation of these equations indicates that they have properties very similar to differential–algebraic equations (DAEs) [7–13]. If in system (1.1), $K(t, s) \equiv 0$, then we have a linear DAE system. In other words, we can consider linear DAEs as a special form of IDAEs (1.1). Also, it can be shown that IDAE system has properties similar to integral-algebraic equations (IAEs). Author, refers the interested reader to [14–24] for more research works on the IAE systems.

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The study of implementation of implicit Runge–Kutta methods of Pouzet-type for singular systems of Volterra integro-differential equations has been provided by Kauthen [25]. He studied the convergence properties of this numerical method as fully discretized collocation method. Brunner [14] analyzed global and local superconvergence of piecewise polynomial collocation solutions for the semi-explicit index-1 IDAEs. Bulatov [26,27] considered a class of first-order integro-differential equations with a degenerate matrix multiplying the derivative and suggested a numerical solution method based on Euler's implicit method together with a quadrature formula using left rectangles. Also the multistep method to solving a certain class of IDAEs has been provided by Bulatov and Chistyakova [28].

The present paper is devoted to the study of numerical solvability of the integro-differential–algebraic equations (1.1). For this aim, in Section 2, using the υ-smoothing property of a Volterra integral operator and some projectors, we decouple the IDAEs (1.1) into the mixed system of Volterra integro-differential equations (VIDEs) and integral equations (VIEs), and then investigate the existence and uniqueness solution of the obtained system. In Section 3, the operational Tau method as well-known method is applied to approximate the solution of IDAE system (1.1). Convergence analysis of the proposed numerical method is investigated in Section 4 and in Section 5, the results of numerical experiments are compared with analytical solution and with those of other recently published methods to confirm the accuracy and efficiency of the new scheme which is presented in this paper.

2. Existence and uniqueness solution

In this section, we firstly use the υ-smoothing property of a Volterra integral operator and some projectors to decouple the system (1.1) into the inherent system of Volterra integro-differential equations (VIDEs) and a system of Volterra integral equations (VIEs), and then the construction of solution of the system (1.1) is introduced by the existence and uniqueness theorem.

Definition 1 (*[20]*). The Volterra integral operator in (1.1) corresponding to the kernel matrix $K(t,s) = \begin{pmatrix} k_{pq}(t,s) \\ p,q=1,\ldots,d \end{pmatrix}$, with $d \geq 2$, is said to be υ-smoothing if there exist integers $\upsilon_{pq} \geq 1$ with $\upsilon = \max_{1\leq p,q\leq d}\{\upsilon_{pq}\}$ such that the following conditions hold:

(1) $\frac{\partial^i k_{pq}(t,s)}{\partial t^i}|_{s=t} = 0,\ t \in \Omega,\ i = 0,\ldots,\upsilon_{pq}-2,$
(2) $\frac{\partial^{\upsilon_{pq}-1} k_{pq}(t,s)}{\partial t^{\upsilon_{pq}-1}}|_{s=t} \neq 0,\ t \in \Omega,$
(3) $\frac{\partial^{\upsilon_{pq}} k_{pq}(t,s)}{\partial t^{\upsilon_{pq}}} \in C(D).$

We set $\upsilon_{pq} = 0$ when $k_{pq}(t,s) \equiv 0$.

Now, let the Volterra integral operator in (1.1) be 1-smoothing and $K = K(t,t)$, we rewrite system (1.1) as:

$$A(t)(P(t)X(t))' + B_1(t)(P(t)X(t) + Q(t)X(t)) + V + W = f(t), \tag{2.1}$$

where $Q(t)$ denotes a projector onto $ker\,A(t)$, $P(t) = I - Q(t)$, $Q(t)^2 = Q(t)$, $B_1(t) = B(t) - A(t)P'(t)$ and

$$V = \int_0^t K(t,s)P(s)X(s)ds, \qquad W = \int_0^t K(t,s)Q(s)X(s)ds.$$

Also, system (2.1) can be rewritten as

$$A_1(t)\big(P(t)(P(t)X(t))' + Q(t)X(t)\big) + B_1(t)P(t)X(t) + V + W - KQ(t)X(t) = f(t), \tag{2.2}$$

where $A_1(t) = A(t) + B_1(t)Q(t) + KQ(t)$. Let $u = P(t)X(t),\ v = Q(t)X(t)$ and $det(A_1(t)) \neq 0,\ \forall t \in \Omega$. Multiplying (2.2) by $P(t)A_1^{-1}(t)$ and $Q(t)A_1^{-1}(t)$, respectively, we have the following mixed system of Volterra integro-differential equations and Volterra integral equations

$$\begin{cases} u' + B_2(t)u + B_3(t)v + \int_0^t \bar{K}(t,s)u(s)ds + \int_0^t \bar{K}(t,s)v(s)ds = g_1(t), \\ B_4(t)v + B_5(t)u + \int_0^t \hat{K}(t,s)u(s)ds + \int_0^t \hat{K}(t,s)v(s)ds = g_2(t), \end{cases} \tag{2.3}$$

where $B_2(t) = \Big(P(t)A_1^{-1}(t)B_1(t) - Q(t)P'(t)\Big)$, $B_3(t) = \Big(-P(t)A_1^{-1}(t)K - Q(t)P'(t)\Big)$, $B_4(t) = \Big(I - Q(t)A_1^{-1}(t)K\Big)$, $B_5(t) = Q(t)A_1^{-1}(t)B_1(t)$, $\bar{K}(t,s) = P(t)A_1^{-1}(t)K(t,s)$, $\hat{K}(t,s) = Q(t)A_1^{-1}(t)K(t,s)$, $g_1(t) = P(t)A_1^{-1}(t)f(t)$ and $g_2(t) = Q(t)A_1^{-1}(t)f(t)$.

If $B_4(t)$ be invertible, then differentiating from the second equation of (2.3) and inserting u' from the its first equation and some manipulations, lead to the following second-kind integro-differential equation

$$v' + \tilde{B}_6(t)u + \tilde{B}_7(t)v + \int_0^t \tilde{K}(t,s)u(s)ds + \int_0^t \tilde{K}(t,s)v(s)ds = \tilde{g}_2(t), \tag{2.4}$$

where the meaning of $\tilde{B}_6(t)$, $\tilde{B}_7(t)$, $\tilde{K}(t,s)$ and $\tilde{g}_2(t)$ is clear.