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Nonconforming generalized multiscale finite element methods

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HIGHLIGHTS

- We propose a framework for the nonconforming generalized multiscale methods with explicit steps concretely given to illustrate our methodology.
- Our approach is simple and noble in the sense that it is free of parameters which are essential in other existing discontinuous Galerkin methods.
- Oversampling ideas enhance approximation quality of our approach.
- Our method performs well to resolve highly heterogeneous media.
- We provide several numerical examples to confirm the efficiency of the proposed method.

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ABSTRACT

A framework is introduced for nonconforming multiscale approach based on GMsFEM (Generalized Multiscale Finite Element Method). Snapshot spaces are constructed for each macro-scale block. The snapshot spaces can be based on either conforming or nonconforming elements. With suitable dimension reduction, offline spaces are constructed. Moment function spaces are then introduced to impose continuity among the local offline spaces, which results in nonconforming GMsFE spaces. Oversampling and randomized boundary condition strategies are considered. Steps for the nonconforming GMsFEM are given explicitly. Error estimates are derived. Numerical results are presented to support the efficiency of the proposed approach.

1. Introduction

Many processes have multiscale nature. Typical examples include groundwater transport in subsurface formation and composite materials. Due to scale disparity, some type of model reduction is necessary. There are many multiscale approaches, *e.g.*, numerical homogenization [1–4], multiscale finite element methods [5–7], variational multiscale methods [8], multiscale finite volume methods [9] and heterogeneous multiscale methods [10,11]. Each of the above mentioned papers proposed its own framework for the derivation of multiscale finite element method, while the common goal is to construct multiscale basis functions.

For the sake of simple explanation, consider the problem, given $f \in H^{-1}(\Omega)$, to find $p \in H_0^1(\Omega)$ such that

$$\mathscr{L}p = f, \quad \mathscr{L} = -\nabla \cdot \kappa \nabla : H_0^1(\Omega) \to H^{-1}(\Omega).$$

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In multiscale methods, one can find a local approximation of the solution in each given block. From these local solutions, we typically construct multiscale basis functions $\{\phi_j\}_j$ and seek the solution in the form

$$p = \sum_j p_j \phi_j$$

by coupling multiscale basis functions via a global formulation [5]. This coupling typically requires some overlapping for conforming approaches or some parameters, which may increase the complexity. It is desired to couple local solutions with as minimal information as possible and keeping a very few DOFs in each macro-scale block (cf. [12]). Here, we follow the nonconforming approach [13–16] and present a general framework for nonconforming multiscale finite element methods. The proposed framework can also be considered as a contribution towards the development of general nonconforming finite element concepts.

Traditional nonconforming methods use piecewise polynomials as approximation space. These piecewise polynomials need to satisfy some continuity requirements across inter-element boundaries in order to ensure a convergent method. For example, the well-known Crouzeix–Raviart element [13] on a triangular mesh uses P_1 (piecewise linear) polynomials in each triangle, then the basis functions are required to be continuous at the midpoint of each edge. Instead of polynomials, the nonconforming generalized multiscale finite element methods use spectral basis functions as local approximation space (see also [6] for spectral multiscale DG approach). These local basis functions are then glued together by some analogous continuity conditions.

The idea of using a nonconforming approach has been employed to the multiscale methods as early as in 2010 by Efendiev et al. [17]. It was proposed that local problems are solved in oversampled domains, and the multiscale shape functions are obtained by restricting the solutions of the local problems to the respective macro-scale elements. Such restrictions normally introduce discontinuity across macro-scale elements, hence the method is nonconforming. Recently, the nonconforming MsFEMs have been developed further by utilizing the Crouzeix–Raviart elements for solving the diffusion equation [18], the advection–diffusion problem [19], and the Stokes equations [20] in perforated domains or domains with periodic coefficients. Associated with each interface between macro-scale elements, one multiscale basis function is constructed. The construction leads to natural boundary conditions for the multiscale basis functions which relaxes the sensitivity of the methods in highly non-periodic settings without using the oversampling technique.

Our nonconforming multiscale approach is based on the generalized multiscale finite element method (GMsFEM) framework [21], which constructs enriched nonconforming multiscale function spaces. As discussed in [21], such an enrichment is in general needed for problems with complex heterogeneity. In the GMsFEM, the snapshot spaces and the offline spaces are constructed in each macro-scale block. First, the snapshot spaces are designed to represent the solution in each macro-scale block and they are constructed, typically, by solving a number of local problems. From these snapshot spaces, a suitable dimension reduction is applied to build offline spaces. These form our multiscale space, which is used to solve the global problem. In our nonconforming GMsFEM framework, we propose to develop moment spaces as well as local approximation spaces. Both of the spaces are important for the accuracy of the proposed method. The construction of both spaces begins with a snapshot space construction and then an appropriate local spectral decomposition is used to identify multiscale basis functions. We note that the moment functions are used only in the offline stage to impose "the continuity" in the multiscale space. This is very different from other approaches such as DG (discontinuous Galerkin methods) and HDG (hybridized discontinuous Galerkin methods), where some carefully chosen parameters are used in the online stage.

In this paper, we also consider oversampling strategies, which improve the accuracy of multiscale methods. The main idea of oversampling is to solve local boundary value problems in larger regions and then restrict the solutions to the original regions in constructing multiscale basis functions [22,7,23]. We propose the use of oversampling approaches in constructing both moment and basis functions.

To reduce the cost for the construction of the moment space, we use a randomized boundary condition technique. Instead of solving local problems corresponding to all possible boundary conditions on the micro-scale mesh, we solve local problems corresponding to only a few random boundary conditions. This can reduce the computation cost for obtaining the nonconforming GMsFE space while the approximation property of the underlying space can be maintained.

We present some representative numerical results. High contrast and periodic media are considered. The effects of local enrichment, oversampling, and randomized boundary condition technique are demonstrated through different numerical examples. In all cases, we observe convergence as we increase the number of basis functions. It should be remarked that the nonconforming multiscale finite element spaces constructed using the oversampling technique perform always better.

The rest of the paper is organized as follows. In Section 2, we state the model problem as well as the discrete problems in micro-scale and macro-scale meshes. Then the detailed construction of the nonconforming multiscale finite element space is discussed in Section 3 and an oversampling strategy is described in Section 4. We analyze the approximation errors and consistency errors of our proposed method in Section 5, followed by some representative numerical results in Section 6. In Section 7, we conclude the paper.

2. A framework for nonconforming generalized multiscale finite element methods

In this section a nonconforming approach to design a generalized multiscale finite element method is presented for elliptic problems. This framework can be used to solve several other multiscale problems, such as elasticity equations,