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On the construction of trivariate near-best quasi-interpolants based on C^2 quartic splines on type-6 tetrahedral partitions

D. Barrera [a], M.J. Ibáñez [a,*], S. Remogna [b]

[a] Department of Applied Mathematics, University of Granada, Campus de Fuentenueva s/n, 18071-Granada, Spain

[b] Department of Mathematics, University of Torino, via C. Alberto, 10, 10123 Torino, Italy

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ABSTRACT

The construction of new quasi-interpolants (QIs) having optimal approximation order and small infinity norm and based on a trivariate C^2 quartic box spline is addressed in this paper. These quasi-interpolants, called near-best QIs, are obtained in order to be exact on the space of cubic polynomials and to minimize an upper bound of their infinity norm which depends on a finite number of free parameters in a tetrahedral sequence defining the coefficients of the QIs. We show that this problem has always a unique solution, which is explicitly given. We also prove that the sequence of the resulting near-best quasi-interpolants converges in the infinity norm to the Schoenberg operator.

1. Introduction

The construction of appropriate non-discrete models from given discrete volume data is an important problem in many applications, such as scientific visualization, computer graphics, medical imaging, numerical simulation, etc.

Classical approaches are based on trivariate tensor-product polynomial splines. If we require a certain smoothness along the coordinate axes, such splines can be of high coordinate degree, that can create unwanted oscillations and often require (approximate) derivative data at certain prescribed points. These reasons raise the natural problem of constructing alternative smooth spline models, that use only data values on the volumetric grid and simultaneously approximate smooth functions as well as their derivatives. Moreover, in order to avoid unwanted oscillations, it is desirable that polynomial sections have total degree, instead of the coordinate degree, that is typical of tensor product schemes.

Therefore, in the literature, alternative smooth spline models using only data values on the volumetric grid and of total degree have been proposed. A first possible approach, beyond the classical tensor product scheme, is represented by blending sums of univariate and bivariate C^1 quadratic spline quasi-interpolants (see e.g. [1–3]). Other methods based on trivariate C^1 splines of total degree have been proposed, in [4–6] on type-6 tetrahedral partitions, in [7] on truncated octahedral partitions, in [8–10] on Powell–Sabin (Worsey–Piper) split, and in [11] by using quadratic trivariate super splines on uniform tetrahedral partitions. Furthermore, higher smoothness C^2 has been considered in [12–14,1,15], where the reconstruction of volume data is provided in the space of C^2 quartic splines.

The aim of this paper is to continue the investigation of such kind of trivariate spaces of total degree four and smoothness C^2, with approximation order four. In particular, we propose the construction of a new general family of quasi-interpolants (abbr. QIs) on $\mathbb{R}^3$, called of near-best type, motivated by the good results obtained by this method in the univariate and bivariate settings (see [16–22,1,23,24]).

* Corresponding author.
E-mail addresses: dbarrera@ugr.es (D. Barrera), mibanez@ugr.es (M.J. Ibáñez), sara.remogna@unito.it (S. Remogna).

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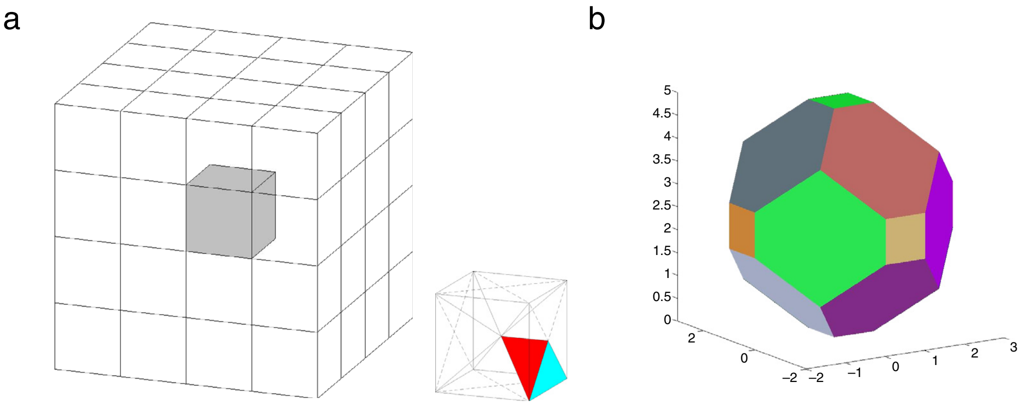

Fig. 1. (a) The uniform type-6 tetrahedral partition and (b) the support of the seven directional box spline.

Moreover, we recall that a fundamental property of quasi-interpolants is that they do not require the solution of huge systems of linear equations, as occurs in the construction of interpolating operators, and this is very important in the 3D setting.

In particular, this general family of operators is constructed by imposing the exactness on the space $\mathbb{P}_3$ of trivariate polynomials of total degree at most three and by minimizing an upper bound for the operator infinity norm.

Such a technique has been partially used in [14] for the construction of QIs on a bounded domain based on C^2 quartic splines. Since the main goal is to deal with functions defined on a bounded domain, it is necessary to construct coefficient functionals associated with boundary generators (i.e. generators with supports not completely inside the domain), so that the functionals involve data points inside or on the boundary of the domain. Therefore, they propose to minimize an upper bound for the infinity norm of the operator, by using a technique that takes into account both the value of such an upper bound and the position of the data points.

The main difference with respect to the present paper is that in this paper we propose and analyse the general construction of near-best QIs in the whole space $\mathbb{R}^3$.

The paper is organized as follows. In Section 2, we recall definitions and properties of the space of C^2 quartic splines on type-6 tetrahedral partitions. In Section 3, we explain in details the construction of near-best QIs. They are obtained by solving a minimization problem that admits always a unique solution. We provide norm and error estimates. In Section 4 we provide some results concerning the performances of the near-best QIs when the degree of the involved box spline increases. Finally, a section devoted to conclusions is included.

2. On the space of trivariate C^2 quartic splines

In this section we study the spline space generated by the integer translates of a trivariate C^2 quartic box spline specified by a set of seven directions.

We consider the box spline proposed in [25], that is a box spline whose direction vectors form a cube and its four diagonals, thus $\mathbb{R}^3$ is cut into a symmetric regular arrangement of tetrahedra called type-6 tetrahedral partition (see Fig. 1(a)).

Following [25], we consider the set of seven direction vectors of $\mathbb{Z}^3$ and spanning $\mathbb{R}^3$

$$X = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$$

defined by

$$\begin{aligned} &e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1), \quad e_4 = (1, 1, 1), \\ &e_5 = (-1, 1, 1), \quad e_6 = (1, -1, 1), \quad e_7 = (-1, -1, 1). \end{aligned} \tag{2.1}$$

According to [26, Chap. 11] and [27, Chap. 1], since the set X has seven elements and the domain is $\mathbb{R}^3$, the box spline $B(\cdot) = B(\cdot|X)$ is of degree four. The continuity of the resulting box spline depends on the determination of the number d, such that $d + 1$ is the minimal number of directions that needs to be removed from X to obtain a reduced set that does not span $\mathbb{R}^3$: then one deduces that the continuity class is C^{d-1}. In our case $d = 3$, thus the polynomial pieces defined over each tetrahedron are of degree four and they are joined with C^2 smoothness.

The support of the C^2 quartic box spline B is the truncated rhombic dodecahedron centred at the point $(\frac{1}{2}, \frac{1}{2}, \frac{5}{2})$ and contained in the cube $[-2, 3] \times [-2, 3] \times [0, 5]$, see Fig. 1(b). Its projections on the coordinate planes are the octagonal supports of the bivariate C^2 quartic box spline with the following set of directions of $\mathbb{R}^2$: $\{(1, 0); (0, 1); (1, 1); (1, 1); (-1, 1); (-1, 1)\}$.