Journal of Computational and Applied Mathematics 311 (2017) 262–271

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Reproducing kernel method for the numerical solution of the Brinkman–Forchheimer momentum equation

S. Abbasbandy [a,*], R.A. Van Gorder [b], P. Bakhtiari [a]

[a] Department of Mathematics, Imam Khomeini International University, Qazvin, 34149-16818, Iran

[b] Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG, United Kingdom

ARTICLE INFO

Article history:
Received 23 June 2015
Received in revised form 15 July 2016

MSC:
35J40
46E22
65N12

Keywords:
Brinkman–Forchheimer momentum equation
Reproducing kernel method
Boundary value problem
Convergence analysis

ABSTRACT

We consider two efficient methods for the solution of the Brinkman–Forchheimer momentum equation with boundary conditions on the square. Physically, this model describes the flow of fully developed forced convection in a porous-saturated rectangular duct. After first demonstrating the existence and symmetry properties of a solution, we apply the reproducing kernel method in order to solve the Brinkman–Forchheimer momentum equation. We then demonstrate the applicability of the method by considering several specific numerical examples, which allow us to understand the variation of the physical solutions as one changes any of the several model parameters. The numerical results demonstrate the utility of the reproducing kernel method for solving nonlinear elliptic partial differential equations on compact domains.

1. Introduction

The problem of fully developed forced convection in a porous-saturated parallel plate channel, with the inclusion of boundary and inertial effects, was considered analytically and numerically by Hooman [1]. In particular, Hooman [1] considered the Brinkman–Forchheimer momentum equation over one variable and obtained a perturbation solution for forced convection in a saturated porous duct. Such a result gives qualitative information about the fully developed forced convection through a porous medium bounded by two isoflux parallel plates under the Brinkman–Forchheimer model. Weidman and Medina [2] studied the linear Brinkman equation in one dimension. Regarding a spatial formulation over multiple variables, Hooman and Merrikh [3] studied a linear form of the partial differential equation over two variables, and this approximates the flow within a duct in the limit where nonlinear effects due to inertia and density are negligibly small. Recently, in [4], the Brinkman–Forchheimer equation for a two-dimensional cross section, particularly a rectangular duct, was studied via a variational approximation method. In that work, the nonlinear terms studied in [1] are maintained, and some qualitative results are given for the relevant two-dimensional flow. Therefore, the study [4] can be seen as the spatial generalization of the model studied in [1] to the case of a duct.

The Brinkman–Forchheimer momentum equation for a unidirectional flow in the (orthogonal) z direction of a square duct is given (over the square domain $\Phi = [0, 1] \times [0, 1]$) by

$$\tilde{\theta}\left(\frac{\partial^2}{\partial x^2}u(x, y) + \frac{\partial^2}{\partial y^2}u(x, y)\right) - \frac{C_f\rho}{\sqrt{K}}u^2(x, y) - \frac{\theta}{K}u(x, y) + G = 0, \tag{1}$$

* Corresponding author.
E-mail address: abbasbandy@yahoo.com (S. Abbasbandy).

http://dx.doi.org/10.1016/j.cam.2016.07.030

where $u(x, y)$ is the velocity in the z direction, C_f is the inertial coefficient, ρ is the density, θ is the viscosity of the fluid, $\tilde{\theta}$ is the effective viscosity, K is the permeability, and G is the adverse applied pressure gradient (see [5,4] and references therein). The appropriate boundary conditions are

$$u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0, \tag{2}$$

which simply means that there is no slip at the surface of the rectangular duct. Hence, the velocity profile should be zero at the boundary, which makes complete physical sense. Defining the constants a, b, c as

$$a \equiv \frac{C_f \rho}{\tilde{\theta}\sqrt{K}}, \qquad b \equiv \frac{\theta}{\tilde{\theta} K}, \qquad c \equiv \frac{G}{\tilde{\theta}}, \tag{3}$$

and the two-dimensional Laplacian operator as

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{4}$$

we can write the partial differential equation (1) and boundary conditions (2) in the compact form

$$\Delta u = au^2(x, y) + bu(x, y) - c, \tag{5}$$

$$u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0. \tag{6}$$

Throughout the present paper, we shall be concerned with the numerical solution of the elliptic boundary value problem (5)–(6).

Although the concept of a reproducing kernel was introduced by Zaremba in 1908 [6] in harmonic functions with boundary conditions, it was not formally studied until Bergman attempted to provide a fundamental framework for its theory during the forth decade in the 20th century [7,8]. In 1950, Aronszajn produced a systematic reproducing kernel space method based on the Bergman's works [9,10]. Since then, many researchers in various fields of science and engineering have used the reproducing kernel space method [11–22]. Based on our best knowledge, Cui and Lin are pioneers in linear and nonlinear numerical analysis using the reproducing kernel method (RKM) [23–25].

Generally speaking, the aim of the RKM is to compute the approximate solution of a given equation from an infinite-dimensional Sobolev space by means of simpler functions, generally coming from a finite-dimensional space. From the point of view of numerical analysis, RKM enjoys at least the following three substantial benefits. At first, it is possible to get arbitrary good approximations to the solution of a given equation by making the dimension of the approximating space sufficiently large. Secondly, the elements of the approximating space is simple, so that they can be easily integrated or differentiated. Finally, there exists a well-developed theory to facilitate the analysis of the resulting computational procedures. Polynomials and splines (piecewise polynomials) are in an ideal choice on all three quantities.

The rest of this paper is organized as follows. In Section 2, we demonstrate that solutions to (5)–(6) do exist, and that such solutions will exhibit symmetry. We use this later point later when displaying cross sections of the numerical results, in order to qualitatively understand the influence of the model parameters on the solutions. In Section 3, we formulate the proper RKM to obtain the numerical solution of the Brinkman–Forchheimer momentum equation. Specific numerical simulations are given in Section 4 in order to demonstrate the accuracy and utility of the numerical approach. We also use such results in order to study the dependence of the solutions of the Brinkman–Forchheimer momentum equation on the model parameters. Concluding remarks are given in Section 5.

2. Existence and symmetry of solutions

In this section, we shall briefly demonstrate the existence and symmetry properties of solutions to (5)–(6).

Theorem 2.1. *Consider the nonlinear Laplace equation $-\Delta u = g(u)$ on the domain $\Phi = [0, 1] \times [0, 1]$, subject to homogeneous boundary data $u = 0$ on the boundary $\partial\Phi$. We assume $u \in C^2(\Phi)$. Let $g(u) = -au^2 - bu + c$ with $a, b, c \geq 0$, so that we recover the equation of interest. Then, a non-negative solution u exists provided that there exists some function $w \geq 0$ such that $-\Delta w \geq g(w)$ in Φ and $w \geq 0$ on the boundary $\partial\Phi$.*

Proof. Pick the function $w(x, y) = x^2 + y^2 + \sqrt{5 + c}$. Clearly, $w > 0$ on the boundary $\partial\Phi$. The remaining condition $-\Delta w \geq g(w)$ becomes

$$-4 \geq -a(x^2 + y^2)^2 - 2a\sqrt{5 + c}(x^2 + y^2) - 5 - c - b(x^2 + y^2) + c,$$

which is equivalent to

$$1 \geq -a(x^2 + y^2)^2 - (2a\sqrt{5 + c} + b)(x^2 + y^2).$$

Now, $-a(x^2 + y^2)^2 - (2a\sqrt{5 + c} + b)(x^2 + y^2) \leq 0$ for all $a, b, c \geq 0$, so the inequality is always true. Hence, the function $w(x, y) = x^2 + y^2 + \sqrt{5 + c}$ satisfies $-\Delta w \geq g(w)$ in Φ. We therefore meet the hypotheses of the main theorem of [26], and the needed result holds.