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Comonotonic approximations of risk measures for variable annuity guaranteed benefits with dynamic policyholder behavior

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ABSTRACT

The computation of various risk metrics is essential to the quantitative risk management of variable annuity guaranteed benefits. The current market practice of Monte Carlo simulation often requires intensive computations, which can be very costly for insurance companies to implement and take so much time that they cannot obtain information and take actions in a timely manner. In an attempt to find low-cost and efficient alternatives, we explore the techniques of comonotonic bounds to produce closed-form approximation of risk measures for variable annuity guaranteed benefits. The techniques are further developed in this paper to address in a systematic way risk measures for death benefits with the consideration of dynamic policyholder behavior, which involves very complex path-dependent structures. In several numerical examples, the method of comonotonic approximation is shown to run several thousand times faster than simulations with only minor compromise of accuracy.

1. Introduction

Quantitative modeling, pricing and risk management of variable annuities have become an active area of research, driven by rapid market innovation and increasing complexity of guaranteed benefits. Non-traditional quantitative techniques are required for quantifying, assessing and managing embedded option-like investment features. In recent years, regulators in North American markets have set up capital requirement standards for equity-linked insurance products based on Monte Carlo simulations. Among their many great advantages, simulation methods are known for their universal applications to complex systems of product designs and their easy implementation, especially with the rapid improvement of computational power. Bauer et al. [1] and Bacinello et al. [2] gave comprehensive treatments of major product designs of guaranteed benefits by simulations. However, one should bear in mind that simulation-based techniques are sampling procedures that provide statistical estimations. It is a well-known fact that the sampling error of Monte Carlo simulation in general decreases by $1/\sqrt{n}$ with n being the sample size. In other words, the sample size has to increase a hundredfold in order for the estimate to improve one significant digit. Many industrial surveys, such as Farr et al. [3], have reported the growing problems of inefficient simulation exercises which make it extremely difficult to obtain useful information and make decisions on pricing and risk management in a timely manner. It is not surprising that practitioners often have to strike a difficult balance between the accuracy of results and the efficiency of their simulations.

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There has been growing interest in both the industry and the literature for the improvement of model efficiency by either analytical methods or statistical means. For example, Koursaris [4] discussed the computation of capital requirement by least squares Monte Carlo simulations. Bauer et al. [5] compared least squares Monte Carlo simulations and numerical PDEs for valuing surrender options in equity-linked insurance.

The pricing of various types of variable annuity guaranteed benefits is extensively studied in the actuarial literature. Hardy [6] provided a comprehensive review of option pricing theory and its applications to many investment guarantees. Ulm [7] and Ulm [8] derived analytical solutions to guaranteed minimum death benefits (GMDB) with rollup and ratchet options;Chi and Lin [9] introduced a PDE method for pricing guaranteed minimum maturity benefit (GMMB) and GMDB with continuously paying premiums. As an alternative, a closed-form approximation for the same guarantees with flexible premium payments was derived in Costible [10]. Marshall et al. [11] studied the valuation of a guaranteed minimum income benefit (GMIB). Bernard et al. [12] proposed models for optimal surrender strategy for various guaranteed benefits with surrender options. However, less is known with regard to the risk management of these guaranteed benefits. For many complex product designs, Monte Carlo simulations remain the only available tool for computing risk measures. Nevertheless, efforts have been made in the recent literature to draw analytical techniques non-conventional to actuarial literature to the computation of risk measures. Feng and Volkmer [13] developed integral solutions to risk measures of GMMB and GMDB net liabilities using Yor's representation of the joint distribution of geometric Brownian motion and its time-integral. An improvement using spectral expansion techniques was made in Feng and Volkmer [14].

Variable annuities are financial contracts between annuity writers (typically life insurers) and individual policyholders. Policyholders make purchase payments into investment accounts at the inception and expect to reap financial gain on the investment of their accounts. Let us first consider the cash flows of a stand-alone variable annuity contract. The life cycle of a variable annuity contract can be broken down into two phases. The first is known as the *accumulation phase*, in which policyholders' investment accounts grow in proportion to certain equity-indices in which policyholders choose to invest at the inception. Let $\{S_t, 0 \le t \le T\}$ describe the dynamics of the underlying equity-index from the inception of the contract to the maturity T (which is assumed to be an integer) and $\{F_t, 0 \le t \le T\}$ describe the evolution of fund values in a particular policyholder's investment account with F_0 being the initial purchase payment. Let us consider the discrete time model with a valuation period of $1/n$ of a time unit, i.e. $t = 1/n, 2/n, \ldots, k/n, \ldots, T$. The fees and charges by annuity writers are typically taken as a fixed percentage of the-then-current account values on a periodic basis. The equity-linked mechanism for variable annuity dictates that

$$F_{k/n} = F_0 \frac{S_{k/n}}{S_0} \left(1 - \frac{m}{n}\right)^k, \quad k = 1, 2, \ldots, nT,$$

where m is the annual rate of total charge compounded n times per year, and charges are made at the beginning of each valuation period. This annual charge m is also referred to in practice as the mortality and expense (M&E) fee. Let r be the continuously compounding yield rate per year on bonds backing up the guaranteed benefits. Observe that the income from the insurer's perspective is generated by a stream of account-value-based payments. The present value of fee incomes, also called margin offset, up to the kth valuation period is given by

$$M_{k/n} = \sum_{j=0}^{k-1} e^{-rj/n} \left(\frac{m_e}{n}\right) F_{j/n}$$

where m_e is the annual rate of GMMB rider charge compounded n times each year (part of the total charge m allocated to fund the GMMB). Although fee incomes $\{M_t, t \ge 0\}$ are considered separately as incoming cash flows, the fee rate m_e is included in the M&E fee rate m, as m is always greater than m_e to allow for overheads, commissions and other expenses.

The second phase typically starts at the beginning of payments from guaranteed benefits and is called the *income phase*. The models of the liabilities differ greatly by the designs of investment guarantee. In this paper, we consider the two most common types of benefits.

Guaranteed minimum maturity benefit (GMMB)—individual model

In the case of a GMMB, the policyholder is guaranteed to receive a minimum balance G in the investment account at maturity T, if he/she survives maturity. The present value of the gross liability to the insurer is

$$e^{-rT}(G - F_T)_+ I(T_x > T),$$

where $(x)_+ = \max\{x, 0\}$ and T_x is the future lifetime of the policyholder of age x at inception. Consider the net liability of the guaranteed benefits from the insurer's perspective, which is the gross liability of guaranteed benefits in the income phase less the fee incomes in the accumulation phase. The present value of the GMMB net liability is given by

$$L_e^{(n)}(T_x) := e^{-rT}(G - F_T)_+ I(T_x > T) - \sum_{j=0}^{(nT \wedge T_x)-1} e^{-rj/n} \left(\frac{m_e}{n}\right) F_{j/n},$$

where $x \wedge y = \min\{x, y\}$.