Journal of Computational and Applied Mathematics 311 (2017) 306–313

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Sharp numerical inclusion of the best constant for embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ on bounded convex domain

Kazuaki Tanaka [a,*], Kouta Sekine [b], Makoto Mizuguchi [a], Shin'ichi Oishi [b]

[a] *Graduate School of Fundamental Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan*

[b] *Faculty of Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan*

A R T I C L E I N F O

Article history:
Received 2 December 2015
Received in revised form 18 July 2016

Keywords:
Computer-assisted proof
Elliptic problem
Embedding constant
Error bounds
Sobolev inequality
Verified numerical computation

A B S T R A C T

In this paper, we propose a verified numerical method for obtaining a sharp inclusion of the best constant for the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ on a bounded convex domain in $\mathbb{R}^2$. We estimate the best constant by computing the corresponding extremal function using a verified numerical computation. Verified numerical inclusions of the best constant on a square domain are presented.

1. Introduction

We consider the best constant for the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, i.e., the smallest constant $C_p(\Omega)$ that satisfies

$$\|u\|_{L^p(\Omega)} \le C_p(\Omega) \|u\|_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega), \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ $(n = 2, 3, \ldots)$, $2 < p < \infty$ if $n = 2$, and $2 < p \le 2n/(n-2)$ if $n \ge 3$. Here, $L^p(\Omega)$ $(1 \le p < \infty)$ is the functional space of pth power Lebesgue integrable functions over Ω. Moreover, assuming that $H^1(\Omega)$ denotes the first order L^2-Sobolev space on Ω, we define $H_0^1(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega \text{ in the trace sense}\}$ with inner product $(\cdot,\cdot)_{H_0^1(\Omega)} := (\nabla\cdot, \nabla\cdot)_{L^2(\Omega)}$ and norm $\|\cdot\|_{H_0^1(\Omega)} := \|\nabla\cdot\|_{L^2(\Omega)}$.

Such constants are important in studies on partial differential equations (PDEs). In particular, our interest is in the applicability of these constants to verified numerical computation methods for PDEs, which originate from Nakao's [1] and Plum's work [2] and have been further developed by many researchers. Such methods require explicit bounds for the embedding constant corresponding to a target equation at various points within them (see, e.g., [3–7]). Moreover, the precision in evaluating the embedding constants directly affects the precision of the verification results for the target equation. Occasionally, rough estimates of the embedding constants lead to failure in the verification. Therefore, accurately estimating such embedding constants is essential.

It is well known that the best constant in the classical Sobolev inequality has been proposed [8,9] (see Theorem A.1). A rough upper bound of $C_p(\Omega)$ for a bounded domain $\Omega \subset \mathbb{R}^n$ can be obtained from the best constant by considering zero extension outside Ω (see Corollary A.2). Moreover, Plum [6] proposed another estimation formula that requires not the boundedness of Ω but an explicit lower bound for the minimum eigenvalue of $-\Delta$ (see Theorem A.3), where Δ denotes

* Corresponding author.
E-mail address: imahazimari@fuji.waseda.jp (K. Tanaka).

http://dx.doi.org/10.1016/j.cam.2016.07.021
0377-0427/

the usual Laplace operator. Although these formulas enable us to easily compute the upper bound of $C_p(\Omega)$, little is known about the best constant.

In this paper, we propose a numerical method for obtaining a verified sharp inclusion of the best constant $C_p(\Omega)$ that satisfies (1) for a bounded convex domain $\Omega \subset \mathbb{R}^2$. As a verified result, we prove the following theorem by using our method through a computer-assisted technique:

Theorem 1.1. *For the square $\Omega_s = (0, 1)^2$, the smallest values of $C_p(\Omega_s)$ $(p = 3, 4, 5, 6, 7)$ that satisfy (1) are enclosed as follows:*

$$C_3(\Omega_s) \in [0.25712475017618,\ 0.25712475017620];$$
$$C_4(\Omega_s) \in [0.28524446071925,\ 0.28524446071929];$$
$$C_5(\Omega_s) \in [0.31058015094505,\ 0.31058015094512];$$
$$C_6(\Omega_s) \in [0.33384042151102,\ 0.33384042151112];$$
$$C_7(\Omega_s) \in [0.35547994288611,\ 0.35547994288634].$$

Remark 1.2. Since it follows from a simple variable transformation that

$$C_p((a,b)^2) = (b-a)^{\frac{2}{p}} C_p(\Omega_s), \tag{2}$$

the values in Theorem 1.1 can be directly used for all squares $(a, b)^2$ $(-\infty < a < b < \infty)$ by multiplying them with $(b-a)^{2/p}$. Moreover, these values can be applied to deriving an explicit upper bound of $C_p(\Omega)$ for a general domain $\Omega \subset (a, b)^2$ by considering zero extension outside Ω, while the precision of the upper bound depends on the shape of Ω.

Hereafter, we replace the notation $C_p(\Omega)$ with $C_{p+1}(\Omega)$ $(1 < p < \infty$ if $n = 2$, and $1 < p \le (n+2)/(n-2)$ if $n \ge 3)$ for the sake of convenience. The smallest value of $C_{p+1}(\Omega)$ can be written as

$$C_{p+1}(\Omega) = \sup_{u \in H_0^1(\Omega)\setminus\{0\}} \Phi(u), \tag{3}$$

where $\Phi(u) = \|u\|_{L^{p+1}(\Omega)} / \|u\|_{H_0^1(\Omega)}$. This variational problem is still the topic of current research (see, e.g., [10,11] and the references therein).

The boundedness of $C_{p+1}(\Omega)$ in (3) is ensured by considering zero extension outside Ω (see Corollary A.2). In addition, it is true that the supremum $C_{p+1}(\Omega)$ in (3) can be realized by an extremal function in $H_0^1(\Omega)$. A proof of this fact is sketched as follows. Let $\{u_i\} \in H_0^1(\Omega)$ be a sequence such that $\|u_i\|_{H_0^1(\Omega)} = 1$ and $\|u_i\|_{L^{p+1}(\Omega)} \to C_{p+1}(\Omega)$ as $i \to \infty$. The Rellich–Kondrachov compactness theorem (see, e.g., [12, Theorem 7.22]) ensures that there exists a subsequence $\{u_{i_j}\}$ that converges to some u^* in $L^{p+1}(\Omega)$. Moreover, there exists a subsequence $\{u_{i_k}\} \subset \{u_{i_j}\}$ that converges to some $u' \in H_0^1(\Omega)$ in the weak topology of $H_0^1(\Omega)$ because $H_0^1(\Omega)$ is a Hilbert space. Since $\{u_{i_k}\}$ converges to u^* in $L^{p+1}(\Omega)$, it follows that $u^* = u'$. Hence, $u^* \in H_0^1(\Omega)(\subset L_{p+1}(\Omega))$ and $\|u^*\|_{L^{p+1}(\Omega)} = C_{p+1}(\Omega)$.

Since $|u| \in H^1(\Omega)$ for all $u \in H^1(\Omega)$ (see, e.g., [12, Lemma 7.6]) and $\Phi(u^*) = \Phi(|u^*|)$, we are looking for the extremal function u^* such that $u^* \ge 0$ (in fact, the later discussion additionally proves that $u^* > 0$ in Ω). The Euler–Lagrange equation for the variational problem is

$$\begin{cases} -\Delta u = lu^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

with some positive constant l (see, e.g., [10] for a detailed proof). Since Φ is scale-invariant (i.e., $\Phi(ku^*) = \Phi(u^*)$ for any $k > 0$), it suffices to consider the case that $l = 1$ for finding an extremal function u^* of Φ (recall that we consider the case that $p > 1$). Moreover, the strong maximum principle ensures that nontrivial solutions u to (4) such that $u \ge 0$ in Ω are positive in Ω. Therefore, in order to find the extremal function u^*, we consider the problem of finding weak solutions to the following problem:

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

This problem has a unique solution if $\Omega \subset \mathbb{R}^2$ is bounded and convex [13]. Therefore, we can obtain an inclusion of $C_{p+1}(\Omega)$ as $\|u^*\|_{L^{p+1}(\Omega)} / \|u^*\|_{H_0^1(\Omega)}$ by enclosing the solution u^* to (5) with verification.

Numerous numerical methods for verifying a solution to semilinear elliptic boundary value problems exist (e.g., [3–7] along with related works [14,15]). Such methods enable a concrete ball containing exact solutions to elliptic equations to be obtained; this is typically in the sense of the norms $\|\cdot\|_{H_0^1(\Omega)}$ and $\|\cdot\|_{L^\infty(\Omega)}$, where $L^\infty(\Omega)$ is the functional space of Lebesgue