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A new reproducing kernel method for variable order fractional boundary value problems for functional differential equations

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ABSTRACT

Based on reproducing kernel theory, a numerical method is proposed for solving variable order fractional boundary value problems for functional differential equations. In the previous works, piecewise polynomial reproducing kernels were employed to solve fractional differential equations. However, the computational cost of fractional order operator acting on such kernel functions is high. In this paper, reproducing kernels with polynomial form will be constructed and applied to solve variable order fractional functional boundary value problems. The method can reduce computation cost and provide highly accurate global approximate solutions.

1. Introduction

Recently, variable fractional differential equations have been used to modeling signal processing, the processing of geographical data and signature verification. The study of such problems has attracted much attention. Razminia, Dizaji and Majd [1] proved the existence of the solution for a generalized fractional differential equation with non-autonomous variable order operators. Sun and Chen [2–5] introduced some important applications of variable fractional derivative.

Due to the existence of variable fractional derivative, it is usually impossible to obtain the analytical solution of such equations. Hence, we must find numerical methods for solving such problems. Fu, Chen and Ling [6] applied the method of approximate particular solutions to both constant- and variable-order time fractional diffusion models. Liu, Shen, Zhang et al. [7–15] proposed various finite difference methods for variable order fractional partial diffusion equations. Sierociuk, Malesza and Macias [16] introduced a numerical scheme for a variable order derivative based on matrix approach. Yu and Ertürk [17] applied a finite difference method to variable order fractional integro-differential equations. Zhao, Sun and Karniadakis [18] derived two second-order approximation formulas for the variable-order fractional time derivatives. Zayernouri and Karniadakis [19] developed fractional spectral collocation methods for linear and nonlinear variable order fractional partial differential equations. Combining Legendre wavelets functions and operational matrices, Chen, Wei et al. [20] presented a numerical method to solve a class of nonlinear variable order fractional differential equations. Atangana [21] gave the Crank–Nicholson scheme for time-fractional variable order telegraph equation. Li and Wu [22] proposed a reproducing kernel method for variable fractional boundary value problems.

Reproducing kernel theory plays an important role in various fields of mathematics, such as probability and statistics, and operator theory. Recently, based on the reproducing kernel theory, a method called the reproducing kernel method

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for solving operator equations was proposed by Cui, Geng et al. [23,24]. Recently, the method has been used and modified by many authors [25–32]. In this paper, we will construct reproducing kernels with polynomial form and apply it to the following variable fractional functional boundary value problems

$$\begin{cases} D^{\alpha(x)}u + a(x)u'(x) + b(x)u(x) + c(x)u(\tau(x)) = f(x), & -1 < x < 1, \\ u(-1) = \mu_1, \qquad u(1) = \mu_2 \end{cases} \tag{1.1}$$

where $1 < \alpha(x) \le 2$, $-1 \le \tau(x) \le 1$, μ_1, μ_2 are constants, $a(x),\ b(x),\ c(x),\ \tau(x) \in C^2[-1, 1]$, $D^{\alpha(x)}$ denotes the variable order Caputo fractional derivative defined as follows

$$D^{\alpha(x)}u(x) = \frac{1}{\Gamma(2-\alpha(x))}\int_{-1}^{x}(x-t)^{1-\alpha(x)}u''(t)dt. \tag{1.2}$$

Note here that $\tau(x)$ may be larger or smaller than x.

2. Construction of reproducing kernels with polynomial form

Denote by P_n the set of all algebraic polynomials of degree $\le n$, namely,

$$P_n := span\{1, x, x^2, \dots, x^n\},$$

equipped with the following inner product and norm

$$(u(y), v(y)) = \int_{-1}^{1} u(y)v(y)\omega(y)dy \tag{2.1}$$

and

$$\|u\| = \sqrt{(u, u)},$$

where ω is a generic weight function.

We say that a finite or infinite sequence of vectors $f_1, f_2, \dots$ in an inner product space is orthogonal if

$$(f_i, f_j) = 0, \quad (i \ne j).$$

Theorem 2.1. *For any given positive weight function $\omega \in L^1(-1, 1)$, the sequence of polynomials defined as follows is orthogonal:*

$$p_i(x) = (x - a_i)p_{i-1}(x) - b_i p_{i-2}(x), \quad (i \ge 2) \tag{2.2}$$

with $p_0(x) = 1,\ p_1(x) = x - a_1$, and

$$a_i = \frac{(xp_{i-1}, p_{i-1})}{(p_{i-1}, p_{i-1})}, \qquad b_i = \frac{(xp_{i-1}, p_{i-2})}{(p_{i-2}, p_{i-2})}.$$

Proof. We proceed with the proof by using an induction argument. We show by induction on n that $(p_n, p_i) = 0$ for $i = 0, 1, \dots, n-1$.

For $n = 1$,

$$(p_1, p_0) = ((x - a_1)p_0, p_0) = (xp + 0, p_0) - a_1(p_0, p_0).$$

From the definition of a_1, we have $(p_1, p_0) = 0$.

Now we assume the validity of our assertion for $n-1$, $(n \ge 2)$. Then

$$(p_n, p_{n-1}) = (xp_{n-1}, p_{n-1}) - a_n(p_{n-1}, p_{n-1}) - b_n(p_{n-2}, p_{n-1}) = 0$$

and

$$(p_n, p_{n-2}) = (xp_{n-1}, p_{n-2}) - a_n(p_{n-1}, p_{n-2}) - b_n(p_{n-2}, p_{n-2}) = 0.$$

For any $i \le n-3$, one obtains

$$\begin{aligned}(p_n, p_i) &= (xp_{n-1}, p_i) - a_n(p_{n-1}, p_i) - b_n(p_{n-2}, p_i) \\ &= (p_{n-1}, xp_i) \\ &= (p_{n-1}, p_{i+1} + a_{i+1}p_i + b_{i+1}p_{i-1}) = 0\end{aligned}$$

and the proof is complete. □