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Energy method for structure-preserving finite difference schemes and some properties of difference quotient

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ABSTRACT

In this article we introduce the energy method for structure-preserving finite difference schemes which inherit the physical structures such as energy conservation or dissipative laws. Another aim is to give some useful properties for difference quotient which is compatible with the structure-preserving finite difference schemes. The method and properties enable us not only to take the problem with more general nonlinearity but also to improve proofs of error estimate between the numerical and exact solutions. Lastly we give two examples of application to the Cahn–Hilliard and the Boussinesq type equations.

1. Introduction

Numerical schemes which inherit physical structures for the original problem in some sense are called *structure-preserving numerical schemes*. There are several structure-preserving numerical methods. One of examples is symplectic integrator which preserves geometrical quantity from the viewpoint of dynamical system theory (see [1–3] etc.). However, throughout this article we use the term "structure-preserving" in the sense that the schemes inherit the physical properties such as energy conservation laws and entropy increasing laws. When we study numerical analysis for nonlinear partial differential equations, the structure-preserving numerical methods give us a lot of benefits. One of the merits is that the stability of solution is often satisfied automatically. The observation to the conserved or decreasing quantity plays an role as checker whether the computer program includes human errors or not, which is also another merit. At these reasons many researchers study the structure-preserving numerical schemes (see e.g. [4–11] and reference therein). Recently several ways to derive structure-preserving schemes are proposed. Furihata in [6] proposed the *discrete variational derivative method* (DVDM) which enables us to derive the structure-preserving scheme systematically. In [12], the another method called *average vector field method* was proposed as another method to derive structure-preserving schemes. Our methods introduced in this article can be applied to these numerical schemes.

These physical structures such as conservation and dissipative laws also play important roles in theory of partial differential equations (PDEs). For example, it is classical and standard procedure that we construct a local in time solution in the energy class and extend it to global in time solution with the help of the energy conservation law (see e.g. [13]). The energy class means the function space naturally defined from the energy. Let us call the procedure the *energy method*.

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The main contribution of this article is to introduce the energy method for the structure-preserving finite difference schemes. Structure-preserving finite difference schemes are often implicit and nonlinear, and hence even existence of solution for the scheme is not trivial to say nothing of error estimate. Our strategy is as follows, we first show the unique existence of energy class solution for the scheme at time step $(n+1)$ for given solution at n. After that, with the help of a priori estimate which is shown by structure-preserving properties, we show the existence of solution for every time step by inductive argument. In the previous existence results, the local solution was constructed in a different topology from energy class, but to do so it often requires more restrictive assumption. Indeed, in the case of the Cahn–Hilliard equation in [6], to show the existence the assumption $\Delta t/\Delta x^2 \ll 1$ is needed. On the other hand, as we shall show in Section 3, by using the energy method the assumption is replaced by $\Delta t \ll 1$.

Thus main problem is how to establish the local existence. In general the energy class is weaker space than the space that the classical solution belongs to. Thus the construction of time local solution often causes difficulty in PDE theory (see [14,15]). To apply the energy laws to non-smooth solution such as energy class solution, we have to justify the formal calculation in derivation of the laws under smoothness assumption of solution by using some kind of convergence analysis (see also [16]). On the other hand, in discrete case we do not need such a justification as we shall introduce in Section 2. This is because all the finite dimensional normed spaces are equivalent to each other. This method can be applied to equations which have appropriate a priori estimate and have local solution in the class corresponding to the a priori estimate. We will introduce several applications here, and another application to certain thermoelastic system is found in [17]. However, we remark that it does not seem to be easy to judge whether local solution can be constructed or not, as PDEs.

Another aim of this article is to give properties for difference quotient defined in (2.21) given later. For example, for the structure-preserving numerical scheme of the Cahn–Hilliard equation given in [6] the nonlinear term is approximated by the concept. Like this, the difference quotient often appears when we consider structure-preserving numerical schemes (see e.g. [12,8,10]). Nevertheless its mathematical treatment for general nonlinearity without polynomial seems to be few as far as the author knows. Therefore, we give a symmetric identity for the difference quotient (Proposition 2.5) and some estimates for difference quotient (Lemmas 2.4 and 2.6). These properties enable us to prove existence of approximate solution for the scheme with nonlinearity of not only polynomial type. Moreover, these properties are also utilized for an error estimate.

The rest of this article is organized as follows. Main results of this article are gathered in Section 2. After we set up notation and present some lemmas used later, we shall introduce the energy method for the structure-preserving finite difference schemes using artificial semilinear heat equation as an example. Some properties for difference quotient will be given next. Section 3 is devoted to give applications to the Cahn–Hilliard equation and the Boussinesq type problem.

2. Main results

In this section we first set up notation, terminology and several fundamental lemmas which will be used later. Next we introduce the energy method by using certain artificial problem. Lastly we give several estimates and an identity for the difference quotient.

2.1. Preliminaries

For simplicity, we restrict to the problem in one-space dimension case. Let us consider the problem in space–time domain $[0, L] \times [0, T] (\ni (x, t))$. Let us define $C^m(\Omega)$ as the function space of m-times continuous differentiable functions on $\Omega \subset \mathbb{R}$. We remark that the domain Ω will be used in various situations, for instance, in some place as a subset of $[0, L]$ or in other place as a bounded ball $\{\xi \mid |\xi| \leq R\}$, and so on. We also use the notation such as $C^m(\Omega \times [0, T])$ which means the function space of m-times continuous differentiable functions with respect to both space and time variables. We denote partial differential operators with respect to space variable x and time variable t by ∂_x and ∂_t, and similarly we define the differential operators with respect to ξ, η, γ by $\partial_\xi, \partial_\eta, \partial_\gamma$ respectively. In particular, in the case of single variable function we may also denote the derivatives such as F', F'' and F'''. Let K and N be any natural numbers. We split space interval $[0, L]$ into Kth parts and time interval $[0, T]$ into Nth parts with space and time mesh sizes Δx and Δt, and hence the following relations hold $L = K\Delta x$ and $T = N\Delta t$. In the finite difference method we pursue values at $(k\Delta x, n\Delta t)$ with $k = 0, 1, \dots, K$ and $n = 0, 1, \dots, N$. We use a notation such as $f_k^{(n)}$ as the value at $(k\Delta x, n\Delta t)$. We also use expression in bold print to denote vectors with respect to space variable such as $\boldsymbol{f}^{(n)} := (f_k^{(n)})_{k=0}^K$ and especially for single variable case $\boldsymbol{f} := (f_k)_{k=0}^K$. For the approximation to derivatives and integral, we follow the notation of [8], namely, the difference operators δ_n^+, δ_k^+, δ_k^-, $\delta_k^{\langle 1\rangle}$ and $\delta_k^{\langle 2\rangle}$ are defined by

$$
\begin{aligned}
&\delta_n^+ f_k^{(n)} := \frac{f_k^{(n+1)} - f_k^{(n)}}{\Delta t}, \qquad \delta_k^+ f_k^{(n)} := \frac{f_{k+1}^{(n)} - f_k^{(n)}}{\Delta x}, \qquad \delta_k^- f_k^{(n)} := \frac{f_k^{(n)} - f_{k-1}^{(n)}}{\Delta x}, \\
&\delta_k^{\langle 1\rangle} f_k^{(n)} := \frac{f_{k+1}^{(n)} - f_{k-1}^{(n)}}{2\Delta x}, \qquad \delta_k^{\langle 2\rangle} f_k^{(n)} := \frac{f_{k+1}^{(n)} - 2f_k^{(n)} + f_{k-1}^{(n)}}{\Delta x^2},
\end{aligned}
\tag{2.1}
$$