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On the approximation power of generalized T-splines

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ABSTRACT

The paper presents some properties of Generalized T-splines (GT-splines), which are crucial to their actual application. In particular, we construct a dual basis for a noteworthy class of GT-splines, which allows to show that, under suitable conditions, they form a partition of unity. Moreover, we study the approximation properties of the GT-spline space by constructing a class of quasi-interpolants which belong to it and are defined by giving a dual basis.

1. Introduction

The introduction of the concept of T-mesh in the recent years was a significant advancement for the use of multivariate spline functions. The basic idea behind this approach was first introduced in [1], where the authors propose to construct spline surfaces using control points not necessarily lying, topologically, on a regular rectangular grid whose edges always intersect at "cross junctions", allowing to have instead partial rows of control point lying on a grid whose vertices can intersect at "T-junctions". This scheme gives several advantages, such as the possibility to locally refine the surfaces, a considerable reduction of the quantity of control points needed, the ability to easily avoid gaps between surfaces to be joined (see, e.g., [1–3]).

While these ideas have been applied mainly to polynomial splines (see, e.g., [4–9]), recently they were generalized [10,11] to the noteworthy non-polynomial case of the generalized B-splines (GB-splines). The GB-splines are a particularly relevant class of non-polynomial splines, thanks to their adaptability and their applications like the isogeometric analysis (see, e.g., [12–14]): in short, the GB-splines are basis of spaces of piecewise functions, locally spanned both by polynomials and by two other suitable functions. In [10], the T-spline approach was applied to the trigonometric GB-splines, and the basic properties and the linear independence of the obtained blending functions were studied. In [11] the results given in [10] were further generalized to any kind of GB-spline to define the Generalized T-splines, and the class of VMCR T-meshes, guaranteeing the linear independence of the associated T-splines and GT-splines, was introduced.

The goal of this paper is to study some additional properties of the GT-splines, which are crucial for their future practical use, especially considering that the polynomial T-splines are already employed in applications (see, e.g., [15]). In particular, we will show that, under certain conditions, it is possible to prove that the GT-splines form a partition of unity. This results,

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similarly to what was done in [4], will be achieved by constructing a suitable set of dual functionals. Moreover, this set of functionals will be used to construct a class of quasi-interpolant operators, which, in turn, will allow us to get some results about the approximation power of the space spanned by the GT-splines. In particular, in order to guarantee the full approximation order of the GT-spline space, it is crucial that the norms of the elements of the dual basis do not diverge as the T-mesh is refined. We will show in detail that, at least in some noteworthy cases, such bound can be obtained.

Section 2 contains the definition and main properties of univariate GB-splines, including an overview about the construction of their dual basis and a noteworthy example where the norms of the elements of the dual basis are bounded. In Section 3 we recall the notations and some results about GT-splines, needed in the following. In Section 4 we present the main results of the paper, that is, the construction of the dual basis for GT-splines and the approximation power of the GT-spline space by studying a quasi-interpolation projection operator. We will discuss some detailed examples where the norms of the elements of the dual basis can be bounded independently of the refinement of the T-mesh and, as a consequence, we can bound the constant involved in the study of the approximation order of the GT-spline space.

2. Univariate generalized B-splines

2.1. Definition and main properties

We recall the main definition and properties of the GB-splines, which can be also found in [11,13,12]. Let $n, p \in \mathbb{N}, p \geq 2$, and let $\mathbf{\Sigma} = \{s_1 \leq \cdots \leq s_{n+p}\}$ be a non-decreasing knot sequence (*knot vector*); we associate to $\mathbf{\Sigma}$ two vectors of functions $\mathbf{\Omega_u} = \{u_1(s), \ldots, u_{n+p-1}(s)\}$ and $\mathbf{\Omega_v} = \{v_1(s), \ldots, v_{n+p-1}(s)\}$, where, for $i = 1, \ldots, n+p-1$, u_i, v_i belong to $C^{p-2}[s_i, s_{i+1}]$ and are such that the space W spanned by the derivatives

$$U_i(s) = \frac{d^{p-2}u_i(s)}{ds^{p-2}}, \qquad V_i(s) = \frac{d^{p-2}v_i(s)}{ds^{p-2}}$$

is a Chebyshev space, that is, the two following conditions are verified (see also [16]):

$$\begin{aligned} &\forall \psi \in W, \text{ if } \psi^{(p-2)}(s_1) = \psi^{(p-2)}(s_2) = 0, \quad s_1, s_2 \in [a,b],\ s_1 \neq s_2 \\ &\qquad \text{then } \psi^{(p-2)}(s) = 0, \quad s \in [a,b]; \end{aligned} \tag{1}$$

$$\begin{aligned} &\forall \psi \in W, \text{ if } \psi^{(p-2)}(s_1) = \psi^{(p-1)}(s_1) = 0, \quad s_1 \in (a,b), \\ &\qquad \text{then } \psi^{(p-2)}(s) = 0, \quad s \in [a,b]. \end{aligned} \tag{2}$$

We remark that the condition (1) is essential to construct the GB-spline functions defined below, while the condition (2) is a key point to prove some approximation properties of the spaces spanned by the GT-splines in Section 4.

We consider the *generalized spline space* of the functions which, restricted to each interval $[s_i, s_{i+1}]$, belong to the space spanned by $\{u_i(s), v_i(s), 1, s, \ldots, s^{p-3}\}$ for $p \geq 3$ and by $\{u_i(s), v_i(s)\}$ for $p = 2$. For the generalized spline space it is possible to define a basis of compactly-supported splines called *Generalized B-splines*.

Having required that the space spanned $W = \langle U_i, V_i \rangle$ is a Chebyshev space, it is not restrictive to choose, as generating functions of W, $U_i(s)$ and $V_i(s)$ such that

$$U_i(s_i) > 0, \qquad U_i(s_{i+1}) = 0, \qquad V_i(s_i) = 0, \qquad V_i(s_{i+1}) > 0. \tag{3}$$

We will call the selected functions $U_i(s)$ and $V_i(s)$ *generating functions associated to* $[s_i, s_{i+1}]$. Following [12,13], we can define a basis of compactly-supported spline functions for the generalized spline space in the following way: for $p = 2$

$$N_i^{(2)}[\mathbf{\Sigma}, \mathbf{\Omega_u}, \mathbf{\Omega_v}](s) = \begin{cases} \dfrac{V_i(s)}{V_i(s_{i+1})}, & \text{if } s_i \leq s < s_{i+1}, \\ \dfrac{U_{i+1}(s)}{U_{i+1}(s_{i+1})}, & \text{if } s_{i+1} \leq s < s_{i+2}, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

while, for $p \geq 3$,

$$N_i^{(p)}[\mathbf{\Sigma}, \mathbf{\Omega_u}, \mathbf{\Omega_v}](s) = \int_{-\infty}^{s} \left(\delta_i^{(p-1)} N_i^{(p-1)}(r) - \delta_{i+1}^{(p-1)} N_{i+1}^{(p-1)}(r)\right) dr, \quad i = 1, \ldots, n, \tag{5}$$

where

$$\delta_i^{(p)} = \left[\int_{-\infty}^{\infty} N_i^{(p)}(r)\,dr\right]^{-1}, \quad i = 1, \ldots, n. \tag{6}$$