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Iterative Galerkin discretizations for strongly monotone problems

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ABSTRACT

In this article we investigate the use of fixed point iterations to solve the Galerkin approximation of strictly monotone problems. As opposed to Newton's method, which requires information from the previous iteration in order to linearize the iteration matrix (and thereby to recompute it) in each step, the alternative method used in this article exploits the monotonicity properties of the problem, and only needs the iteration matrix calculated once for all iterations of the fixed point method. We outline the abstract *a priori* and *a posteriori* analyses for the iteratively obtained solutions, and apply this to a finite element approximation of a second-order elliptic quasilinear boundary value problem. We show both theoretically, as well as numerically, how the number of iterations of the fixed point method can be restricted in dependence of the mesh size, or of the polynomial degree, to obtain optimal convergence. Using the *a posteriori* error analysis we also devise an adaptive algorithm for the generation of a sequence of Galerkin spaces (adaptively refined finite element meshes in the concrete example) to minimize the number of iterations on each space.
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1. Introduction

In this paper we study Galerkin approximations of strictly monotone problems of the form:

$$\text{find } u \in X : \; A(u, v) = 0 \quad \forall v \in X. \tag{1.1}$$

Here, X is a real Hilbert space, with inner product denoted by $(\cdot, \cdot)_X$ and induced norm $\|x\| = \sqrt{(x, x)_X}$. Furthermore, $A : X \times X \to \mathbb{R}$ is a possibly nonlinear form such that, for any $u \in X$, the mapping $v \mapsto A(u, v)$ is linear and bounded. Moreover, we suppose that A satisfies

(P1) the *strong monotonicity* property

$$A(u, u - v) - A(v, u - v) \geq c_0 \|u - v\|_X^2 \quad \forall u, v \in X, \tag{P1}$$

for a constant $c_0 > 0$, and

(P2) the *Lipschitz continuity* condition

$$|A(u, w) - A(v, w)| \leq L \|u - v\|_X \|w\|_X \quad \forall u, v, w \in X, \tag{P2}$$

with a constant $L > 0$.

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Under these assumptions, there exists a unique solution $u \in X$ of the weak formulation (1.1); see, e.g., [1, Theorem 2.H] or [2, §3.3]. In addition, the solution can be obtained as limit of a sequence $u^0, u^1, u^2, \ldots \in X$ resulting from the fixed point iteration

$$(u^n, v)_X = (u^{n-1}, v)_X - \frac{c_0}{L^2} A(u^{n-1}, v) \quad \forall v \in X, \; n \geq 1, \tag{1.2}$$

for an arbitrary initial value $u^0 \in X$. Indeed, defining the contraction constant

$$k = \sqrt{1 - \left(\frac{c_0}{L}\right)^2}, \tag{1.3}$$

there holds the *a priori* bound

$$\|u - u^n\|_X \leq \frac{k^n}{1-k} \|u^0 - u^1\|_X, \quad n \geq 1, \tag{1.4}$$

for the iteration (1.2), i.e., $\|u - u^n\|_X \xrightarrow{n\to\infty} 0$; incidentally, under the given assumptions, this contraction constant is minimal; see, e.g., [2, Theorem 3.3.23].

Restricting the iteration (1.2) to a finite dimensional linear subspace $X_h \subseteq X$, leads to an iterative Galerkin approximation scheme for (1.1). More precisely, we consider, for an initial guess $u_h^0 \in X_h$ and $n \geq 1$, the iteration

$$u_h^n \in X_h : \; (u_h^n, v_h)_X = (u_h^{n-1}, v_h)_X - \frac{c_0}{L^2} A\left(u_h^{n-1}, v_h\right) \quad \forall v_h \in X_h, \tag{1.5}$$

where c_0 and L are the constants from (P1) and (P2), respectively; we note that (P1) and (P2) indeed hold as X_h is a conforming subspace of X. We emphasize that the problem of finding u_h^n from u_h^{n-1} in the iteration scheme (1.5) is *linear* and uniquely solvable. Similarly as for (1.1) and (1.2), the fixed point iteration (1.5) converges to the (unique) solution $u_h \in X_h$ of the Galerkin formulation

$$A(u_h, v_h) = 0 \quad \forall v_h \in X_h. \tag{1.6}$$

Furthermore, we note the *a priori* bound

$$\|u_h - u_h^n\|_X \leq \frac{k^n}{1-k} \|u_h^0 - u_h^1\|_X, \quad n \geq 1, \tag{1.7}$$

analogous to (1.4).

In solving nonlinear differential equations numerically two approaches are commonly employed. Either the nonlinear problem under consideration is discretized by means of a suitable numerical scheme thereby resulting in a (finite-dimensional) nonlinear algebraic system, or the differential equation problem is approximated by a sequence of (locally) linearized problems which are discretized subsequently. The latter approach is attractive from both a computational as well as an analytical view point; indeed, working with a sequence of linear problems allows the application of linear solvers as well as the use of a linear numerical analysis framework (e.g., in deriving error estimates). In the context of fixed point linearizations (1.5) yet another benefit comes into play: solving for u_h^n from u_h^{n-1} involves setting up and inverting a mass matrix on the left-hand side of (1.5). We emphasize that this matrix is the same for all iterations; hence, it only needs to be computed once (on a given Galerkin space).

The idea of approximating nonlinear problems within a *linear* Galerkin framework has been applied in a variety of works. For example, in the work [3], a Kačanov fixed-point iteration, whereby any nonlinear terms are expressed by means of a previously determined approximation, is employed. Furthermore, in the article [4], the authors have considered general linearizations of strongly monotone operators, and have derived computable estimators for the total error (consisting of the linearization error and the Galerkin approximation error), with identifiable components for each of the error sources. A more specific linearization approach for monotone problems, which is based on the Newton method, has been presented in [5]. In a related context linear finite element approximations resulting from adaptive Newton linearization techniques as applied to semilinear problems have been investigated in the papers [6,7]. Finally, we remark that the linear Galerkin approximation approach for nonlinear problems is not only employed for the purpose of obtaining linearized schemes, but also to address the issue of modeling errors in linearized models; see, e.g. [8,9].

The aim of the current paper is to derive *a priori* and *a posteriori* error bounds for the Galerkin iteration method (1.5). Our error estimates are expressed as the summation of the linearization error resulting from the fixed point formulation with the Galerkin approximation error. In particular, based on the *a posteriori* error analysis, we will develop an adaptive solution procedure for the numerical solution of (1.1) that features an appropriate interplay between the fixed point iterations and possible Galerkin space enrichments (e.g., mesh refinements for finite elements); specifically, our scheme selects between these two options depending on whichever constitutes the dominant part of the total error. In this way, we aim to keep the number of fixed point iterations at a minimum in the sense that no unnecessary iterations are performed if they are not expected to contribute a substantial reduction of the error on the actual Galerkin space.

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