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A Newton conditional gradient method for constrained nonlinear systems

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ABSTRACT

In this paper, we consider the problem of solving constrained systems of nonlinear equations. We propose an algorithm based on a combination of Newton and conditional gradient methods, and establish its local convergence analysis. Our analysis is set up by using a majorant condition technique, allowing us to prove, in a unified way, convergence results for two large families of nonlinear functions. The first one includes functions whose derivative satisfies a Hölder-like condition, and the second one consists of a substantial subclass of analytic functions. Some preliminary numerical experiments are reported.

1. Introduction

In this paper, we consider the problem of finding a solution of the constrained system of nonlinear equations

$$F(x) = 0, \quad x \in C, \tag{1}$$

where $F : \Omega \to \mathbb{R}^n$ is a continuously differentiable nonlinear function and $\Omega \subset \mathbb{R}^n$ is an open set containing the nonempty convex compact set C. This problem appears in many application areas such as engineering, chemistry and economy. The constraint set may naturally arise in order to exclude solutions of the model with no physical meaning, or it may be considered artificially due to some knowledge about the problem itself (see, for example, [1–3] and references therein). Different approaches to solve (1) have been proposed in the literature. Many of them are related to the unconstrained case, having as focus the Newton Method whenever applicable. Strategies based on trust-region and gradient methods have also been used; see, for instance, [1,4,2,5,3,6–9].

In this paper, we propose a Newton conditional gradient (Newton-CondG) method for solving (1), which consists of a Newton step followed by a procedure related to the conditional gradient (CondG) method. The procedure plays the role of getting the Newton iterate back to the constraint set in such a way that the fast convergence of the Newton method is maintained. The CondG method (also known as Frank–Wolfe method [10,11]) requires at each iteration the minimization of a linear functional over the feasible constraint set. This requirement is considered relatively simple and can be fulfilled efficiently in many applications. Moreover, depending on the application, linear optimization oracles may provide solutions with specific characteristics leading to important properties such as sparsity and low-rank; see, e.g., [12,13] for a discussion on this subject. Due to these facts and its simplicity, CondG method has recently received a lot of attention from a theoretical and computational point of view; see, for example, [12,14,13,15–17] and references therein. An interesting approach is to combine variants of CondG method with some superior well designed algorithms (see [15,16] for more details). In this sense, our proposed method seems promising.

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We present a local convergence analysis of the Newton-CondG method. More specifically, we provide an estimate of the convergence radius, for which the well-definedness and the convergence of the method are ensured. Furthermore, results on convergence rates of the method are also established. Our analysis is done via the concept of majorant condition which, besides improving the convergence theory, allows to study Newton type methods in a unified way; see, for example, [18–21] and references therein. Thus, our analysis covers two large families of nonlinear functions, namely, one satisfying a Hölder-like condition, which includes functions with Lipschitz derivative, and another one satisfying a Smale condition, which includes a substantial class of analytic functions. Finally, we also present some preliminary numerical experiments showing the efficiency of our method and discuss its performance compared with the constrained dogleg method studied in [22].

This paper is organized as follows. In Section 2, we study a certain scalar sequence generated by a Newton-type method. The Newton-CondG method and its convergence analysis are discussed in Section 3. Section 4 applies our main convergence result for functions satisfying Hölder-like and Smale conditions. In Section 5, we present some preliminary numerical experiments of the proposed method.

Notations and basic assumptions: Throughout this paper, we assume that $F : \Omega \to \mathbb{R}^n$ is a continuously differentiable nonlinear function, where $\Omega \subset \mathbb{R}^n$ is an open set containing a nonempty convex compact set C. The Jacobian matrix of F at $x \in \Omega$ is denoted by $F'(x)$. We also assume that there exists $x_* \in C$ such that $F(x_*) = 0$ and $F'(x_*)$ is nonsingular. Let the inner product and its associated norm in $\mathbb{R}^n$ be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The open ball centered at $a \in \mathbb{R}^n$ and radius $\delta > 0$ is denoted by $B(a, \delta)$. For a given linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, we also use $\| \cdot \|$ to denote its norm, which is defined by $\|T\| := \sup\{\|Tx\|, \ \|x\| \le 1\}$.

2. Preliminary results

Our goal in this section is to study the behavior of a scalar sequence generated by a Newton-type method applied to solve

$$f(t) = 0,$$

where $f : [0, \ R) \to \mathbb{R}$ is a continuously differentiable function such that

h1. $f(0) = 0$ and $f'(0) = -1$;
h2. f' is strictly increasing.

Although **h1** implies that $t_* = 0$ is a solution of the above equation, the convergence properties of this scalar sequence will be directly associated to the sequence generated by the Newton-CondG method. First, consider the scalar ν given by

$$\nu := \sup\{t \in [0, R) : f'(t) < 0\}. \tag{2}$$

Since f' is continuous and $f'(0) = -1$, it follows that $\nu > 0$. Moreover, **h2** implies that $f'(t) < 0$ for all $t \in [0, \nu)$. Hence, the following Newton iteration map for f is well defined:

$$\begin{array}{rcl} n_f : [0, \ \nu) & \to & \mathbb{R} \\ t & \mapsto & t - f(t)/f'(t). \end{array} \tag{3}$$

Let us also consider the scalars λ and ρ such that

$$\lambda \in [0, 1), \quad \rho := \sup\left\{\delta \in (0, \nu) : (1+\lambda)\frac{|n_f(t)|}{t} + \lambda < 1, \ t \in (0, \delta)\right\}. \tag{4}$$

We now present some properties of the Newton iteration map n_f and show that $\rho > 0$.

Proposition 1. *The following statements hold:*

(a) $n_f(t) < 0$ *for all* $t \in \ (0, \ \nu)$;
(b) $\lim_{t\downarrow 0} |n_f(t)|/t = 0$;
(c) *the scalar* ρ *is positive and*

$$0 < (1+\lambda)|n_f(t)| + t\lambda < t, \quad \forall t \in (0, \ \rho). \tag{5}$$

Proof. (a) From condition **h2** we see that f' is strictly increasing in $[0, R)$, in particular, f is strictly convex. Hence, since $\nu \le R$ (see (2)), we obtain $f(0) > f(t) + f'(t)(0 - t)$, for any $t \in (0, \ \nu)$ which, combined with $f(0) = 0$ and $f'(t) < 0$ for any $t \in (0, \nu)$, proves item (a).

(b) In view of item (a) and the fact that $f(0) = 0$, we obtain

$$\frac{|n_f(t)|}{t} = \frac{1}{t}\left(\frac{f(t)}{f'(t)} - t\right) = \frac{1}{f'(t)}\frac{f(t) - f(0)}{t - 0} - 1, \quad \forall t \in (0, \ \nu). \tag{6}$$

As $f'(0) \neq 0$, item (b) follows by taking limit in (6), as $t \downarrow 0$.