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Iteration Functions re-visited

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ABSTRACT

Two classes of Iteration Functions (IFs) are derived in this paper. The first (one-point IFs) was originally derived by Joseph Traub using a different approach to ours (simultaneous IFs). The second is new and is demonstrably shown to be more *informationally efficient* than the first. These IFs apply to polynomials with arbitrary complex coefficients and zeros, which can also be multiple.

1. Introduction

1.1. Context

It is better to start by putting the material in this paper into the context of our latest global algorithm [1, pp. 62–63]. We compute the zeros of arbitrary polynomials in two distinct stages.

1.1.1. Stage 1

In this first stage we systematically search the complex plane for regions containing the zeros of a given polynomial. This search continues until we are satisfied that each region contains a single zero, which may be a multiple zero of the polynomial.

This search stage could be continued until we were satisfied that the centres of our regions were sufficiently accurate approximations to the true values of the zeros. However, the work is computationally intensive, so we switch to Stage two, which offers quicker convergence.

1.1.2. Stage 2

This second stage uses the centres of our regions found in Stage 1 as initial approximations for IFs that converge rapidly to accurate approximations to the true values of the zeros of a polynomial defined over the complex numbers or the real numbers.

It is the derivation of and discussion about these IFs that form the basis of this paper. The IFs are used for computing the zeros of arbitrary polynomials given suitable initial values for which convergence can be achieved. The main justification for this paper is to present our contention that working with multiple zeros is the best approach [2]; simple zeros are just a special case.

The second *raison d'être* is to present some results that were missing from the first named author's recent Ph.D. thesis [3], namely the exact Asymptotic Error Constants (AECs) of the fourth order and fifth order simultaneous IFs presented therein. In addition, the AEC of the third order IF in (A.35) of [3] is corrected.

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The third reason is to present our approach using IFs in *polynomial format*, which emphasises more clearly how the structure of the IFs changes as we build IFs of higher and higher orders, yielding a power series obtained from our given polynomial.

The results of this paper have been incorporated into a revised version of the thesis. This version is available on the web [1]. All references to the thesis are to the web version rather than the original submission.

The remainder of this paper is set out as follows.

1.2. The scheme of things

Next, Section 2 takes the reader through definitions and equations that are used throughout this paper.

This is followed by Section 3 which derives a class of one-point IFs followed by a class of simultaneous IFs.

Next, Section 4 derives the orders of convergence of the various IFs, and especially their AECs. The IFs for polynomials with only simple zeros are easily derived from the IFs for polynomials with multiple zeros.

Next, Section 5 contains some remarks concerning R-order convergence, where applying an IF in a *serial* fashion, rather than in a *parallel* fashion, can improve the order of convergence in certain cases.

An overview of Stage 1 of the algorithm for finding the zeros of a polynomial is given in Section 6 and information about the software and the experiments is given in Section 7.

Finally, Section 8 presents a summary of our conclusions on the work presented in this paper.

2. Preamble

Let $p(z)$ be a polynomial of degree n given by

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n a_0 \neq 0, \tag{1}$$

where the coefficients have complex or real values and the zeros also have complex or real values. In addition, the zeros $\{\alpha_i\}$ can have multiplicities $\{m_i\}$ greater than one, respectively. The polynomial $p(z)$ has N distinct zeros.

Throughout this paper sums and products will be over the range $1, 2, \ldots, N$ ($N \in \mathbb{N}^+$, the set of positive integers) unless stated otherwise, e.g.

$$\begin{aligned} \sum_{i<\nu} a_i &\equiv \sum_{i=1}^{\nu-1} a_i, \\ \prod_{i\neq\nu} b_i &\equiv \prod_{\substack{i=1\\ i\neq\nu}}^{N} b_i. \end{aligned} \tag{2}$$

When $p(z)$ is *monic*, $a_n = 1$, and we have

$$p(z) = \prod_i (z - \alpha_i)^{m_i}. \tag{3}$$

We next define

$$M = \max_i m_i. \tag{4}$$

Let $p(z)$ be defined as in Eq. (1). The following definitions, originally given by Joseph Traub [4, pp. 5–6], will be used subsequently.

$$u(z) = \frac{p(z)}{p'(z)}, \tag{5}$$

which Joseph Traub calls the *normalised* $p(z)$, and

$$A_i(z) = \frac{p^{(i)}(z)}{i!\,p'(z)}, \quad i = 1, 2, \ldots, n, \tag{6}$$

where $p^{(i)}(z)$ is the ith derivative of $p(z)$ and which Joseph Traub calls the *normalised Taylor series coefficient*. Note that $u(z)$ is often referred to as *Newton's correction* [5, p. 85]. For later use it is worth noting that

$$A_i'(z) = (i+1)A_{i+1}(z) - 2A_2(z)A_i(z), \quad i = 1, 2, \ldots, n. \tag{7}$$

The following definitions will also be useful.

$$S_k(z_\nu) = \sum_{i\neq\nu} \frac{m_i}{(z_\nu - \alpha_i)^k}, \quad k = 1, 2, \ldots, N. \tag{8}$$