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A dispersion minimizing compact finite difference scheme for the 2D Helmholtz equation☆

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ABSTRACT

In this paper, we present a dispersion minimizing compact finite difference scheme for solving the 2D Helmholtz equation, which is a fourth-order scheme. The error between the numerical wavenumber and the exact wavenumber is analyzed, and a refined choice strategy based on minimizing the numerical dispersion is proposed for choosing weight parameters. Numerical results are presented to demonstrate the efficiency and accuracy of the compact finite difference scheme with refined parameters.

1. Introduction

In this paper, we consider the 2D Helmholtz equation

$$\mathcal{L}u := \Delta u + k^2 u = g \tag{1.1}$$

with the wavenumber k, where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the 2D Laplacian, unknown u usually represents a pressure field in the frequency domain, and g denotes the source function. The Helmholtz equation has important applications in acoustic and electromagnetic waves. Obtaining an efficient and more accurate numerical solution for the Helmholtz equation has always been a hot topic in wave computation (see [1–6] and the reference therein).

Solving the Helmholtz equation numerically with high wavenumbers is still a challenging task in the field of computational mathematics [2]. For large wavenumbers, as the solution of the Helmholtz equation oscillates severely, the quality of the numerical results usually deteriorates as the wavenumber k increases, which is the so-called "pollution effect" of high wavenumbers [2,7]. The pollution effect of high wavenumbers is unavoidable for the 2D and 3D Helmholtz equations [2]. Due to the pollution effect of high wavenumbers, the wavenumber of the numerical solution is different from the one of the exact solution, which is known as "numerical dispersion" [7]. Hence, to discretize the Helmholtz equation, there are mainly two issues one should pay attention to: one is the numerical dispersion which is closely related to the pollution error, while the other is the solver cost. A proper discretization for the Helmholtz equation should satisfy that its numerical dispersion is small enough, while its implementation is easy and solving its corresponding linear systems is cheap.

Due to its simplicity and computational efficiency, the finite difference method is a popular and powerful computational technique for numerical seismic wave propagation modeling (cf. [8–10,3,11,5]). There have been various dispersion

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minimizing second-order schemes developed for solving the Helmholtz equation [9,10,12,3,5]. In 1996, a rotated 9-point finite difference scheme was introduced for the Helmholtz equation in [3]. As the authors gave one group of optimal parameters based on minimizing the numerical dispersion, this scheme was indeed a dispersion minimizing scheme. In this paper, 9 points were used to approximate the term of zero order in the Helmholtz equation. This idea, originally proposed in [4], has been proven to be helpful for suppressing the numerical dispersion. A consistent 9-point finite difference scheme for the Helmholtz equation with the perfectly matched layer (PML) was developed in [9]. The authors also proposed refined choice strategies for choosing optimal parameters of the scheme based on minimizing the numerical dispersion. The improvement of the accuracy and the reduction of the numerical dispersion are significant. The idea of the optimal 9-point finite difference scheme proposed in [9] was then extended to the optimal 27-point finite difference scheme for the 3D Helmholtz equation with PML (see, [10]). To improve the numerical accuracy, high-order accurate compact finite difference schemes were also considered. In [11], Harari and Turkel presented various fourth-order methods for time-harmonic wave propagation. A method, depending on the wave's propagation angle, was proposed in [11] to optimize the parameter appearing in the difference formula. As the propagation angle for the wave is not known before, such an optimal method may not be useful in practice. For the 2D Helmholtz equation with constant wavenumbers, fourth-order compact finite difference schemes were presented on uniform grids in [13]. Additionally, two kinds of fourth-order compact schemes were constructed in [14] and the convergence was also analyzed. Sixth-order compact finite difference schemes for the 3D Helmholtz equation with constant coefficients were discussed in [15]. For the 2D and 3D Helmholtz equation with variable coefficients, we refer the interested readers to [8,6] to get high-order compact schemes.

The main purpose of this paper is to develop a dispersion minimizing compact scheme for the 2D Helmholtz equation, which is fourth-order in accuracy. To this end, both the convergence order and numerical dispersion of the scheme will be considered. In the light of the idea of equation based differencing used in [8,13,15,6], we shall first construct a fourth-order approximation for the Laplacian term. The existing high-order compact finite difference schemes usually use one point to approximate the term of zero order in the Helmholtz equation [14,11,13]. To further suppress the numerical dispersion, 9 points will then be used in this paper to formulate a fourth-order approximation for the term of zero order. Combining the fourth-order approximation for the Laplacian term with that for the term of zero order leads to a compact finite difference scheme for the Helmholtz equation. The resulting scheme has weight parameters, which can be chosen properly to improve the accuracy of the solution of the Helmholtz equation. For assessing the numerical dispersion of this scheme, an analysis for the error between the numerical wavenumber and the exact wavenumber will be presented. Moreover, we will give a convergence analysis of the scheme and show that it enjoys the accuracy of fourth-order. To choose optimal coefficients for finite difference schemes, dispersion minimizing schemes proposed in [9,10,12,3,5] usually computed the normalized numerical phase velocity, and then minimized the error between the normalized numerical phase velocity and one. As the normalized numerical phase velocity for some finite difference schemes may not be easily obtained, we will propose an approach for choosing the optimal coefficients of the resulting scheme. This method is also based on minimizing the numerical dispersion. However, it has the advantage of easy implementation.

This paper is organized as follows. In Section 2, we propose a compact finite difference scheme for the 2D Helmholtz equation with constant wavenumbers, and then provide a convergence analysis to show that the scheme is fourth-order in accuracy. For this compact scheme, we present its dispersion equation, and analyze the error between the numerical wavenumber and the exact wavenumber in Section 3. A refined choice strategy for choosing optimal parameters of the scheme based on minimizing the numerical dispersion is proposed in Section 4. In Section 5, a compact finite difference scheme of fourth-order is presented for the 2D Helmholtz equation with variable wavenumbers. Finally in Section 6, four numerical experiments are given to demonstrate the efficiency and accuracy of the scheme. We show that the proposed scheme not only improves the accuracy but also reduces the numerical dispersion significantly.

2. A compact finite difference scheme for the 2D Helmholtz equation

In this section, we propose a compact finite difference scheme for the 2D Helmholtz equation with constant wavenumbers. A convergence analysis is also provided to show that the scheme enjoys the fourth-order accuracy.

To describe the finite difference scheme, we consider the network of grid points (x_m, y_n), where $x_m := x_0 + (m-1)h$ and $y_n := y_0 + (n-1)h$. Note that the same step size $h := \Delta x = \Delta y$ is used for both variables x and y. For each m and n, we set $u_{m,n} := u\,|_{x=x_m, y=y_n}$ and $k_{m,n} := k\,|_{x=x_m, y=y_n}$. We begin with establishing a fourth-order approximation for the Laplacian term. Let $D_{xx}u$ and $D_{yy}u$ denote the second-order centered-difference approximations for u_{xx} and u_{yy}, respectively. By the Taylor expansion, we have that

$$D_{xx}u + D_{yy}u = \Delta u + \frac{h^2}{12}\left(u_{xxxx} + u_{yyyy}\right) + \mathcal{O}(h^4). \tag{2.1}$$

To achieve a compact algorithm, we need to eliminate u_{xxxx} and u_{yyyy} in the above equation. To this end, we differentiate the Helmholtz equation (1.1) twice with respect to x and get that

$$u_{xxxx} = g_{xx} - u_{yyxx} - k^2 u_{xx}. \tag{2.2}$$

Similar treatment with respect to y leads to

$$u_{yyyy} = g_{yy} - u_{xxyy} - k^2 u_{yy}. \tag{2.3}$$