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Construction of a full row-rank matrix system for multiple scanning directions in discrete tomography

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ABSTRACT

A full row-rank system matrix generated by scans along two directions in discrete tomography was recently studied. In this paper, we generalize the result to multiple directions. Let $A\mathbf{x} = \mathbf{h}$ be a reduced binary linear system generated by scans along three directions. Using geometry, it is shown in this paper that the linearly dependent rows of the system matrix A can be explicitly identified and a full row-rank matrix can be obtained after the removal of those rows. The results could be extended to any number of multiple directions. Therefore, certain software packages requiring a full row-rank system matrix can be adopted to reconstruct an image. Meanwhile, the cost of computation is reduced by using a full row-rank matrix.

1. Introduction

The algebraic reconstruction of an image in discrete tomography from the projections in a few directions involves solving an underdetermined linear system of projection equations

$$M\mathbf{x} = \mathbf{b}, \tag{1}$$

where M is an $m \times n^2$ matrix, $\mathbf{x} \in \mathbb{R}^{n^2}$ a reconstructed image vector for an $n \times n$ 2-dimensional image and $\mathbf{b} \in \mathbb{R}^m$ the projection vector. The projection equations are formulated from projection data based on different models. The strip-based projection model [1–4] formulates projection equations according to the fractional areas where each strip-shaped beam intersects with the rectangular lattice of the image to be reconstructed. Thus, it is more realistic than the line-based projection model used in some applications. The two matrices of systems (1) generated by the two projection models along the same direction set are proven to be row equivalent [4].

The l_1-minimization algorithm has been applied to reconstruct an image from the underdetermined system (1) [5,6]. A full row-rank system reduces the cost for solving the system and it is required for the usage of some current l_1-minimization software packages, such as the l_1-magic and the sparselab software packages [7,8]. However, the matrix M is often row-rank deficient so it is desired to convert the matrix M into a full row-rank matrix.

Consider an $n \times n$ 2-dimensional image f defined on a lattice set

$$\Omega = \{(u, v) | 0 \le u, v \le n-1, u, v \in \mathbb{Z}\}$$

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of n^2 lattice points. Suppose that the n^2 lattice points of Ω are arranged in such an order from bottom to top column-wise and the corresponding vector $\mathbf{x}$ representing the unknown image values is given by

$$\mathbf{x} = [f(0,0)\ \cdots f(0,n-1)\, f(1,0)\cdots f(1,n-1)\ \cdots f(n-1,0)\ \cdots f(n-1,n-1)]^T.$$

In general, a direction for parallel beam scanning can be represented by (q,p) for rays with slope $\frac{p}{q}$, $q>0$ and $\gcd(|p|,q)=1$, in addition to the horizontal and vertical directions.

Projection data obtained by each ray or beam provides one equation in system (1). In the case of one direction $(q,-|p|)$ with a negative slope $-\frac{|p|}{q}$, it is known in [4,9] that M is a $(|p|+q)n$ by n^2 binary matrix with $\mathrm{rank}(M)=(|p|+q)n-|p|\,q$, in the form of

$$M = [M_1\ M_2\ \cdots\ M_n],$$
$$M_i = [\mathbf{m}_1^{(i)}\ \mathbf{m}_2^{(i)}\ \cdots\ \mathbf{m}_n^{(i)}] \in \mathbb{R}^{(p+q)n\times n}, \quad 1\le i\le n,$$

where

$$\mathbf{m}_j^{(i)} = \begin{bmatrix} \mathbf{o}_{(i-1)|p|+(j-1)q} \\ \mathbf{e}_{|p|+q} \\ \mathbf{o}_{(n-i)|p|+(n-j)q} \end{bmatrix}, \quad 1\le i,j\le n.$$

Here $\mathbf{o}_s$ represents the zero vector of dimension s and $\mathbf{e}_{|p|+q}$ represents the first column of the identity matrix of order $|p|+q$. The linear dependence of the rows of M has been studied and can be described with the following result.

Lemma 1 ([10]). *The linearly dependent rows of the matrix M can be precisely located so that the removal of these rows will result in a full row-rank matrix. The rows of M with row indices $i=|p|\,u+qv+1$ for nonnegative integers $u,v\le n-1$ are maximal linearly independent rows of M.*

The terminology of a reduced binary system matrix is naturally introduced for discussing scans along multiple directions.

Definition 1 ([10]). If the rows of M with row indices $i\ne |p|\,u+qv+1$, for any nonnegative integers $u,v\le n-1$, are replaced by zero rows and the corresponding components of $\mathbf{b}$ by zeros, the resultant system,

$$A\mathbf{x} = \mathbf{h}, \tag{2}$$

is called the reduced binary system (RBS) generated along a scanning direction (q,p). The reduced matrix A is called the reduced binary system matrix (RBSM) generated along a scanning direction (q,p).

It is clear that the last $|p|+q-1$ rows of A are zero rows. More properties of the RBSM generated along a single direction are summarized in the following Lemma 2.

Lemma 2 ([10]). *The RBSM A generated along a scanning direction (q,p), where $q>0$, is a binary matrix of dimension $(|p|+q)n\times n^2$ with rank $(|p|+q)n-|p|\,q$, having $|p|\,q$ zero rows. The tth row of A is a zero row if and only if $t\ne |p|\,u+qv+1$ for any nonnegative integers $u,v\le n-1$. Each column of A has exactly one entry $a_{ij}=1$ if and only if $i=|p|\,u+qv+1$ and $j=un+v+1$ if $p<0$, or $j=(n-1-u)n+v+1$ if $p>0$, for some nonnegative integers $u,v\le n-1$.*

For scans along two distinct directions (q_1,p_1) and (q_2,p_2) with $\gcd(|p_s|,q_s)=1$, $q_s>0$, $s=1,2$, let $A_1\mathbf{x}=\mathbf{h}_1$ and $A_2\mathbf{x}=\mathbf{h}_2$ be RBSs generated along the two directions, respectively. System (2) is considered with the formulation of

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \quad \text{and} \quad \mathbf{h} = \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{bmatrix}, \tag{3}$$

where A is a $(|p_1|+q_1+|p_2|+q_2)n$ by n^2 binary matrix with rank $(|p_1|+q_1+|p_2|+q_2)n-(|p_1|+|p_2|)(q_1+q_2)$, having $|p_1|\,q_1+|p_2|\,q_2$ zero rows. For convenience we define a *minimal linearly dependent set* in this context.

Definition 2. A linearly dependent set E is said to be minimal if every proper subset of E is linearly independent.

A full row-rank matrix can be constructed explicitly as summarized in the following lemma.

Lemma 3 ([10]). *For scans along two distinct directions (q_1,p_1) and (q_2,p_2), the nonzero rows of the matrix A are partitioned into $|p_1|\,q_2+|p_2|\,q_1$ minimal linearly dependent sets. The removal of all zero rows and one row from each set will result in a full row-rank matrix.*

It is observed that every nonzero row of the matrix A lies in one and only one minimal linearly dependent set of rows.

In general, for scans along multiple directions $\{(q_s,p_s)\}_{s=1}^k$, the system matrix M consists of k submatrices M_s, $s=1,2,\ldots,k$, generated by the k scanning directions, respectively. The corresponding matrix A is formed by k RBSMs, A_s's,