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Induction hardening of steel with restrained Joule heating and nonlinear law for magnetic induction field: Solvability

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ABSTRACT

We study a coupled system of Maxwell's equations with nonlinear heat equation. We start with derivation of a mathematical model assuming a nonlinear dependency of magnetic field $\boldsymbol{H}$ on magnetic induction $\boldsymbol{B}$. Moreover the electric conductivity σ is supposed to be temperature reliant. The coupling between the thermal and electromagnetic equations is provided via the Joule heating term and σ. Next we use time discretization based on the Rothe's method to provide energy estimates for discretized system. We prove the existence of a weak solution $\{u, \boldsymbol{B}\}$ to this coupled system with controlled Joule heating term.

1. Introduction

Induction heating is a process mostly used in industrial areas to enhance mechanical properties of ferrous components such as hardness, strength and fatigue resistance. The advantage of induction hardening is that it can be applied only on a part of the workpiece (surface layer, pins, gears) to sustain properties of remaining parts. During induction hardening an alternating current is flowing through the copper coil. This generates a rapidly changing magnetic field. Then the workpiece (or its part) is placed within the magnetic field. The alternating field invokes eddy currents in the conductive section. These currents heat up the workpiece until the particular temperature is reached and then the workpiece is immediately quenched with a cool water, which leads to the desired hardening effect. Induction hardening experiments are often very expensive and difficult to realize. Therefore mathematical analysis and numerical simulations contribute significantly to the designing process.

Let us first derive an equation describing the evolution of the magnetic induction field. Assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain occupied by an electromagnetic material. The boundary $\partial\Omega$ is supposed to be $C^{1,1}$. The symbol $\boldsymbol{n}$ is the outer normal vector assigned to $\partial\Omega$. We start our derivation by introducing the classical Maxwell's equations (for reference, see [1])

$$\nabla \times \boldsymbol{H} - \partial_t \boldsymbol{D} = \boldsymbol{J} + \boldsymbol{J}_{app}, \qquad \nabla \cdot \boldsymbol{D} = \rho, \tag{1}$$

$$\partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = \boldsymbol{0}, \tag{2}$$

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where $\boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}, \boldsymbol{E}, \boldsymbol{J}, \boldsymbol{J}_{app}$ and ρ stand for the magnetic field, the electric displacement, the magnetic induction field, the electric field, the total current density, the source current and the density of electrical charge, respectively. The curl operator is expressed as $\nabla\times$, the divergence operator is denoted as $\nabla\cdot$ and ∂_t is short expression for time derivative.

Considering eddy current problems we can neglect the term $\partial_t\boldsymbol{D}$, since its variation in time is insignificant. Furthermore we introduce the linear Ohm's law

$$\boldsymbol{E} = \sigma^{-1}\boldsymbol{J} := \gamma\boldsymbol{J}, \tag{3}$$

where the function σ stands for the electric conductivity and is temperature dependent. The function γ is defined as $\gamma := \sigma^{-1}$ and is bounded and strictly positive i.e. $0 < \gamma_* \le \gamma \le \gamma^* < \infty$, for some positive constants γ_* and γ^*. We assume a non-linear constitutional relation between the magnetic field and the magnetic induction field

$$\boldsymbol{H} := \boldsymbol{H}(\boldsymbol{B}).$$

Using (1), (2) and (3) we conclude that the magnetic induction field $\boldsymbol{B}$ is determined by the solution to the following equation

$$\partial_t\boldsymbol{B} + \nabla\times\gamma(u)\nabla\times\boldsymbol{H}(\boldsymbol{B}) = \nabla\times\gamma(u)\boldsymbol{J}_{app} \quad \text{for a.e. } (\boldsymbol{x}, t) \in \Omega\times(0, T) := Q_T, \tag{4}$$

subject to the initial condition

$$\boldsymbol{B}(0) = \boldsymbol{B}_0, \qquad \nabla\cdot\boldsymbol{B}_0 = 0, \quad \text{for } \boldsymbol{x}\in\Omega, t = 0.$$

Let us note that $\nabla\cdot\boldsymbol{B}_0 = 0$ ensures that $\nabla\cdot\boldsymbol{B}(t) = 0$ for all $t > 0$, which can be easily derived from (2). For ease of explanation we shall consider (natural) homogeneous Dirichlet boundary condition

$$\boldsymbol{n}\times\boldsymbol{H} = \boldsymbol{0}, \qquad \boldsymbol{B}\cdot\boldsymbol{n} = 0 \quad \text{for a.e. } (\boldsymbol{x}, t) \in \partial\Omega\times(0, T). \tag{5}$$

To describe a thermal part of our model we use the nonlinear heat equation. We consider the same domain Ω as before. Then the evolution of temperature u in Q_T is determined as a solution to the

$$\partial_t\theta(u) - \nabla\cdot(\lambda_0\nabla u) = \boldsymbol{J}\cdot\boldsymbol{E} \quad \text{for a.e. } (\boldsymbol{x}, t) \in Q_T, \tag{6}$$

with a known initial state

$$u(0) = u_0 \quad \text{for } \boldsymbol{x}\in\Omega, t = 0,$$

and homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{for a.e. } (\boldsymbol{x}, t) \in \partial\Omega\times(0, T).$$

The function λ_0 is supposed to be positive and bounded i.e. $0 < \lambda_* \le \lambda_0 \le \lambda^* < \infty$, for some positive constants λ_* and λ^*. We require the function θ to fulfill some additional conditions i.e. there exists $\theta_* > 0$ such that

$$\begin{aligned} &\theta(0) = 0,\\ &\theta_* \le \theta'(s),\\ &|\theta(s)| \le C(1 + |s|), \end{aligned} \tag{7}$$

for any $s\in\mathbb{R}$. The source term $\boldsymbol{J}\cdot\boldsymbol{E}$ is so-called Joule heat, using (1) and (3) we can write

$$\boldsymbol{J}\cdot\boldsymbol{E} = \gamma|\boldsymbol{J}|^2 = \gamma\left|\nabla\times\boldsymbol{H}(\boldsymbol{B}) - \boldsymbol{J}_{app}\right|^2. \tag{8}$$

Those two equations (electromagnetic and thermal) are connected via the Joule heating term on the right-hand side in (6) and the term $\gamma(u)$ in (4). The most crucial term to control during mathematical treatment of this system is just Joule heating term, therefore we introduce the cut-off function to deal with this term

$$\mathcal{R}_r(x) := \begin{cases} r & \text{if } x > r,\\ x & \text{if } |x| \le r,\\ -r & \text{if } x < -r, \end{cases} \tag{9}$$

where r is a positive constant. Now, we apply this function on the right-hand side in (6) and analyze the truncated system.

One can wonder whether this truncation is not an artificial intervention only to obtain the desired results, but in many applications the temperature in the workpiece is controlled by the current flowing through induction coil. If the temperature reaches certain degree then the current is switched off. This process is perfectly expressed by the cut-off function, hence it is just a natural step which needs to be done.

For the rest of the paper we will be working with truncated Eq. (6). Thus, for better clarity, we write Eq. (6) with truncated right hand side again. Then the temperature function $u(\boldsymbol{x}, t)$ is determined by the following parabolic equation with appropriate initial and boundary conditions

$$\begin{aligned} &\partial_t\theta(u) - \nabla\cdot(\lambda_0\nabla u) = \mathcal{R}_r\left(\gamma\left|\nabla\times\boldsymbol{H}(\boldsymbol{B}) - \boldsymbol{J}_{app}\right|^2\right) && \text{for a.e. } (\boldsymbol{x}, t)\in Q_T,\\ &u(0) = u_0 && \text{for } \boldsymbol{x}\in\Omega, t = 0,\\ &u = 0 && \text{for a.e. } (\boldsymbol{x}, t) \in \partial\Omega\times(0, T). \end{aligned} \tag{10}$$