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Generalized averaged Szegő quadrature rules

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ABSTRACT

Szegő quadrature rules are commonly applied to integrate periodic functions on the unit circle in the complex plane. However, often it is difficult to determine the quadrature error. Recently, Spalević introduced generalized averaged Gauss quadrature rules for estimating the quadrature error obtained when applying Gauss quadrature over an interval on the real axis. We describe analogous quadrature rules for the unit circle that often yield higher accuracy than Szegő rules using the same moment information and may be used to estimate the error in Szegő quadrature rules.

1. Introduction

We are concerned with quadrature rules for integrals of the form

$$\mathcal{I}(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\mathrm{d}\mu(\theta), \tag{1.1}$$

where the measure $\mu(\theta)$ is nondecreasing and has infinitely many points of increase, and is such that all *moments*

$$\mu_j := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ij\theta}\,\mathrm{d}\mu(\theta), \quad j = 0, \pm 1, \pm 2, \ldots, \tag{1.2}$$

exist and are finite. For notational convenience, we let μ be scaled so that $\mu_0 = 1$. The integrand f is assumed to be continuous on the interval $[-\pi, \pi]$.

Introduce for polynomials g and h the inner product

$$(g, h) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{g(e^{i\theta})}h(e^{i\theta})\mathrm{d}\mu(\theta), \tag{1.3}$$

where $i := \sqrt{-1}$ and the bar denotes complex conjugation. There is an infinite sequence of monic orthogonal polynomials $\{\psi_j\}_{j=0}^{\infty}$ with respect to this inner product. The ψ_j are known as *Szegő polynomials* and satisfy the following recursion relations

$$\psi_0(z) = \psi_0^*(z) = 1, \tag{1.4}$$

$$\psi_j(z) = z\psi_{j-1}(z) + \gamma_j\psi_{j-1}^*(z), \quad j = 1, 2, 3, \ldots, \tag{1.5}$$

$$\psi_j^*(z) = \bar{\gamma}_j z\psi_{j-1}(z) + \psi_{j-1}^*(z), \tag{1.6}$$

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where $\psi_j^*(z) := z^j\bar{\psi}_j(z^{-1})$. Thus, if $\psi_j(z) = \sum_{k=0}^{j} \beta_k^{(j)} z^k$, then $\psi_j^*(z) = \sum_{k=0}^{j} \bar{\beta}_{j-k}^{(j)} z^k$. The ψ_j^* are sometimes referred to as *reversed polynomials*. We have $\psi_j^*(0) = 1$ for all $j \geq 0$ and, therefore, by (1.5), the recursion coefficients satisfy $\gamma_j = \psi_j(0)$, $j = 1, 2, 3, \ldots$. They are known as *Schur parameters* or *reflection coefficients* and satisfy $|\gamma_j| < 1$ for all j. The recursion coefficients can be computed by combining (1.4)–(1.6) with

$$\begin{aligned} \gamma_j &= -(1, z\psi_{j-1})/\delta_{j-1}, \\ \delta_j &= \delta_{j-1}(1 - |\gamma_j|^2), \end{aligned} \quad j = 1, 2, 3, \ldots, \tag{1.7}$$

where $\delta_0 = 1$. Many properties of Szegő polynomials are described in, e.g., [1,2].

We are interested in approximating the integral (1.1) by quadrature rules of the form

$$S_\tau^{(n)}(f) = \sum_{k=1}^{n} \omega_k^{(n)} f(\lambda_k^{(n)}), \quad \omega_k^{(n)} > 0, \; \lambda_k^{(n)} \in \Gamma, \tag{1.8}$$

where $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$ denotes the unit circle in the complex plane. The $\lambda_k^{(n)}$ are nodes and the $\omega_k^{(n)}$ weights of the quadrature rule. The nodes depend on the parameter τ. This will be discussed below.

Let $\Lambda_{-(n-1),n-1}$ denote the set of Laurent polynomials

$$L_{n-1}(z) = \sum_{k=-(n-1)}^{n-1} c_k z^k, \quad c_k \in \mathbb{C}, \tag{1.9}$$

of order at most $n - 1$. The quadrature rule (1.8) is said to be a *Szegő quadrature rule* if

$$S_\tau^{(n)}(p) = \mathcal{I}(p) \quad \forall p \in \Lambda_{-(n-1),n-1}, \tag{1.10}$$

where the integral $\mathcal{I}$ is defined by (1.1). This requirement defines the n-point Szegő rule uniquely up to the location of one node, say $\lambda_1^{(n)}$, which can be chosen arbitrarily on the unit circle; see, e.g., [3,4]. There are no quadrature rules of the form (1.8) that are exact for all Laurent polynomials of order n. The parameter τ will be used to fix the node $\lambda_1^{(n)}$; see Section 2.

Laurent polynomials (1.9) with $z = \exp(i\theta)$, $\theta \in \mathbb{R}$, are trigonometric polynomials in θ. For instance,

$$L_{n-1}(\exp(i\theta)) = a_0 + \sum_{k=1}^{n-1} (a_k \cos(k\theta) + b_k \sin(k\theta)),$$

for appropriate coefficients $a_k, b_k \in \mathbb{C}$. Szegő quadrature rules are quadrature rules for trigonometric polynomials of maximal order. This makes them attractive to use for the integration of periodic functions.

Example 1.1. In the special case when $d\mu(t) = dt$, the moments are given by $\mu_0 = 1$ and $\mu_j = 0$ for $j \geq 1$. All recursion coefficients γ_j vanish and, therefore, $\psi_j(z) = z^j$ for $j = 0, 1, 2, \ldots$. The Szegő rule (1.8) has equidistant nodes $\lambda_k^{(n)} \in \Gamma$ on the unit circle and weights $\omega_k^{(n)} = 1/n$ for $1 \leq k \leq n$. This follows from Proposition 2.1 below. The node $\lambda_1^{(n)} \in \Gamma$ can be chosen arbitrarily. Thus, the Szegő rule is a trapezoidal rule. It is well known that the trapezoidal rule gives high accuracy when applied to the integration of smooth periodic functions. Assume that the integrand f is analytic in the annulus $\{z \in \mathbb{C} : 1/\rho \leq |z| \leq \rho\}$ for some $\rho > 1$. Then

$$|S_\tau^{(n)}(f) - \mathcal{I}(f)| \leq C\rho^{-n}, \tag{1.11}$$

where the constant C can be chosen independently of n; see Henrici [5] or Trefethen and Weideman [6] for details. □

It is the purpose of the present paper to present new quadrature rules for the approximation of integrals of the form (1.1). They are analogues for integration on the unit circle of the generalized averaged Gauss rules proposed by Spalević [7,8]. The new rules use the same moment information as the Szegő rule (1.8) but can yield higher accuracy, because they use more nodes. They also can be used to estimate the quadrature error

$$\mathcal{E}_n(f) := \mathcal{I}(f) - S_\tau^{(n)}(f)$$

in Szegő rules (1.8). Estimation of the size of this error is helpful in determining how large n should be chosen in order to obtain an approximation of desired accuracy.

This paper is organized as follows. Section 2 reviews an approach proposed by Gragg [3] for computing the nodes and weights of an n-node Szegő quadrature rule (1.8) from the recursion coefficients. The latter can be determined from the moments (1.2) by the Levinson or Schur algorithms; see, e.g., [3,9,10] for details on these algorithms. Sections 3 and 4 describe two analogues of the averaged Gauss rules proposed by Spalević [7,8] for integration on (part of) the real axis. The new quadrature rules have $2n$ nodes on the unit circle; the set of quadrature nodes of the rules in Section 3 generally do not contain the nodes of the Szegő rule (1.8) as a subset, while the rules described in Section 4 do. Anti-Szegő quadrature rules, described in [11], furnish another approach to estimate the error in Szegő rules (1.8). They are analogues of the anti-Gauss rules introduced by Laurie [12] for estimating the error in Gauss quadrature rules. We outline anti-Szegő rules in Section 5. Computed examples that illustrate the performance of the generalized averaged quadrature rules and compare them to anti-Szegő rules are presented in Section 6. Concluding remarks can be found in Section 7.