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Improved Schur complement preconditioners for block-Toeplitz systems with small size blocks☆

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ABSTRACT

In this paper, we employ the preconditioned conjugate gradient method with the Improved Schur complement preconditioners for Hermitian positive definite block-Toeplitz systems with small size blocks. Schur complement preconditioners have been proved to be an effective method for such block-Toeplitz systems (Ching et al. 2007). The modification is based on Taylor expansion approximation. We prove that the matrices preconditioned by improved Schur preconditioners have more clustered spectra compared to that of the Schur complement preconditioners. Hence, preconditioned conjugate gradient type methods will converge faster. Numerical examples are given to demonstrate the efficiency of the proposed method.

1. Introduction

In this paper we are interested in solving block Toeplitz system

$$A_{n,m}X = B \tag{1.1}$$

where

$$A_{n,m} = \begin{bmatrix} A_0 & A_{-1} & \cdots & A_{1-n} \\ A_1 & A_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{-1} \\ A_{n-1} & \cdots & A_1 & A_0 \end{bmatrix}.$$

Here $A_{n,m}$ is supposed to be a Hermitian positive matrix. X, B are mn-by-m matrices and each A_j is an m-by-m matrix, where $m \ll n$. It is not hard to see that $A_j = A^*_{-j}$. B^* denotes the conjugate transpose of a generic matrix B. Some applications of this kind of linear systems can be found in different fields of mathematics, physics, scientific computing, pattern recognition and engineering [1–15]. These applications have motivated scientists to develop specifically fast algorithms for solving them.

Preconditioned Conjugate Gradient (PCG) method have been proved to be an effective method for solving block-Toeplitz (BT) linear systems, and some important results can be found in [16–21,18]. One of the key of the PCG is that how to choose a suitable preconditioner, which determine the convergence rate of the method. In [16], Wai-ki Ching, Michael K. Ng and

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You-wei Wen studied block diagonal and Schur complement preconditioners for BT systems, and got some excellent results. We start with the introduction of their work. Firstly partition the BT matrix into 2-by-2 blocks as follows:

$$A_{n,m} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{1.2}$$

Here A_{11} and A_{22} are the principal submatrices of $A_{n,m}$, and they are also BT matrices and have same inner structure as $A_{n,m}$, i.e. $A_{11} = A_{22} = A_{n/2,m}$. Therefore (1.1) can be rewritten as follows:

$$\begin{bmatrix} A_{n/2,m} & A_{12} \\ A_{21} & A_{n/2,m} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{1.3}$$

Now we consider block preconditioner $B_{n,m}$ in the following form:

$$B_{n,m} = \begin{bmatrix} A_{n/2,m} & 0 \\ 0 & A_{n/2,m} \end{bmatrix}.$$

Here, w.l.o.g., we may assume n is even.

To get the Schur complement preconditioners, we firstly explain the Schur complement factorization. From the theory of matrices we know that the Schur complement of the block A_{22} of the matrix $A_{n,m}$ is as follows:

$$S_{n,m} = A_{22} - A_{21}(A_{11})^{-1}A_{12}.$$

Thus the linear system (1.3) can be rewritten as follows:

$$\begin{bmatrix} I & 0 \\ A_{21}(A_{11})^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & S_{n,m} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

The author in [16] approximated $S_{n,m}$ by $A_{22} = A_{n/2,m}$ and define the Schur complement preconditioner of $A_{n,m}$ as:

$$C_{n,m} = \begin{bmatrix} I & 0 \\ A_{21}(A_{11})^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} + A_{21}(A_{11})^{-1}A_{12} \end{bmatrix}.$$

We note that this approximation is actually inexact, which motivated us to find a better preconditioner by a more precise approximation of $S_{n,m}$. In fact, we used the idea of Taylor expansion approximation to achieve this goal. Specifically, for an arbitrary matrix A, its Taylor expansion formula can be written as follows:

$$(I - A)^{-1} = I + A + A^2 + A^3 + A^4 + \cdots.$$

If we only reserve the linear part, then we have:

$$(I - A)^{-1} \approx I + A. \tag{1.4}$$

Here we assume that $I - A$ is invertible. This is equivalent to requiring that the spectral radius of A be less than 1.

The main idea of this paper is based on (1.4), which would lead to a more accurate approximation of $S_{n,m}$. Consequently, we can get an improved Schur complement preconditioners for BT systems with small size blocks. The improvements of our method are reflected in the clustering degree of spectrum of the preconditioned matrix and converge rate of the PCG methods. Numerical results show that improved Schur complement preconditioners are more effective than preconditioners proposed in [16].

In the construction of our preconditioners, just as the procedure in [16], we need to compute the inverse of A_{11}. We utilize the iterative calculation procedure in [16] to accomplish this task. And the Gohberg–Semencul formula will be involved in this process.

The outline of this paper is as follows. In Section 2, we show that how to construct improved Schur complement preconditioners. In Section 3, we analyze the spectra of the preconditioned matrices. Numerical results are given in Section 4 to illustrate the effectiveness of our approach. Finally, conclusions are given in Section 5.

2. Improved Schur complement preconditioners

In this section, we present the improved Schur complement preconditioners for (1.1). First, we present the block Schur complement factorization of (1.2) as follows:

$$\begin{aligned} A_{n,m} &= \begin{bmatrix} I & 0 \\ A_{21}(A_{11})^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}(A_{11})^{-1}A_{12} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ A_{21}(A_{11})^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}(A_{11})^{-1}A_{12} \end{bmatrix} \begin{bmatrix} I & (A_{11})^{-1}A_{12} \\ 0 & I \end{bmatrix} \end{aligned} \tag{2.1}$$