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# On uniqueness of numerical solution of boundary integral equations with 3-times monotone radial kernels

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ABSTRACT

The uniqueness of solution of boundary integral equations (BIEs) is studied here when geometry of boundary and unknown functions are assumed piecewise constant. In fact we will show BIEs with 3-times monotone radial kernels have unique piecewise constant solution. In this paper nonnegative radial function $\mathcal{F}^{\delta_3}$ is introduced which has important contribution in proving the uniqueness. It can be found from the paper if $\delta_3$ is sufficiently small then eigenvalues of the boundary integral operator are bigger than $\mathcal{F}^{\delta_3}/2$. Note that there is a smart relation between $\delta_3$ and boundary discretization which is reported in the paper, clearly. In this article an appropriate constant $c_0$ is found which ensures uniqueness of solution of BIE with logarithmic kernel $\ln(c_0 r)$ as fundamental solution of Laplace equation. As a result, an upper bound for condition number of constant Galerkin BEMs system matrix is obtained when the size of boundary cells decreases. The upper bound found depends on three important issues: geometry of boundary, size of boundary cells and the kernel function. Also non-singular BIEs are proposed which can be used in boundary elements method (BEM) instead of singular ones to solve partial differential equations (PDEs). Then singular boundary integrals are vanished from BEM when the non-singular BIEs are used. Finally some numerical examples are presented which confirm the analytical results.

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## 1. Introduction

Nowadays, the Boundary Elements Method (BEM) has received much attention from researchers. It has become an important technique in the computational solution of a number of physical problems in various fields, such as stress analysis, potential flow, fracture mechanics and acoustics [1]. Also the interested reader can see [2–5] for application on some problems.

### 1.1. Boundary integral equation

Let $\Omega$ be a bounded domain with Lipschitz boundary $\Gamma$, and $\mathcal{L}$ be partial differential operator defined on unknown function $u : \Omega \to \mathbb{R}$ with Dirichlet boundary condition [6,7]. Solving boundary value problem (BVP)

$$\mathcal{L}u(x) = f(x) \quad \text{for } x \in \Omega \quad \text{and} \quad u(x) = \bar{u} \quad \text{for } x \in \Gamma, \tag{1.1}$$

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is our aim. If $K(\|x-y\|)$ be fundamental solution of operator $\mathcal{L}$ [6,7], i.e.

$$\mathcal{L}K(\|x-y\|) = -\delta_{Di}(\|x-y\|), \tag{1.2}$$

when $\delta_{Di}$ is Dirac delta, via Green's identities the BVP (1.1) is replaced by the boundary integral equation (BIE)

$$\int_{\Gamma} K(\|x-y\|)q(y)\, d\Gamma_y = g(x) \quad \text{for } x \in \Gamma, \tag{1.3}$$

when the known function $g$ is evaluated as

$$g(x) = -u(x) \int_{\Omega} \delta_{Di}(\|x-y\|)\, d\Omega_y + \int_{\Gamma} \nabla K(\|x-y\|).n(y)\, u(y)\, d\Gamma_y, \tag{1.4}$$

where $n(y)$ is outward normal vector over the boundary at $y$ and $q$ is flux function, i.e. $q = \nabla u.n$ [7]. If BIE (1.3) has unique solution it can be deduced that BVP (1.1) also has unique solution

$$u(x) = \int_{\Gamma} K(\|x-y\|)q(y)dy - \int_{\Gamma} \nabla K(\|x-y\|).n(y)\, u(y)dy \quad \text{for } x \in \Omega.$$

So uniqueness of solution of BIE (1.3) is an important issue in BEM. In fact BIE makes the basis of the BEM, and this method only transforms the BIE into a linear system of equations. The solvability of this system depends on the condition number of the corresponding system matrix (BEM-matrix), because if the BIE does not have a unique solution then the system of equations in the BEM may not have a unique solution, and BEM-matrix will be ill-conditioned. So, we must study the condition number when the uniqueness of the solution is considered. Unfortunately little attention has been paid to this fundamental issue [8]. For simplicity some engineers use constant boundary elements method for solving BIE (1.3) and they assume $q$ as a piecewise constant function over the boundary [9]. This strategy is named as constant BEM. This assumption enables one to calculate boundary integrals easily and it yields a simple linear system [9]. In this paper the uniqueness of constant BEM is proved and invertibility of the corresponding system matrix is investigated. In fact we will show BIE (1.3) has unique piecewise constant solution when kernel $K$ is a 3-times monotone radial function [10]. Some important and applicable kernels which are used as fundamental solutions in BEM are listed here:

- $-\frac{1}{2\pi}\ln(\lambda\,\|x-y\|)$ for a $\lambda > 0$ (fundamental solution of 2D Laplace equation),
- $\frac{1}{2\pi}K_0(\lambda\|x-y\|)$ for a $\lambda \neq 0$ (fundamental solution of 2D Helmholtz equation),
- $\frac{1}{4\pi}\|x-y\|^{-1}$ (fundamental solution of 3D Laplace equation),
- $\frac{1}{4\pi}\exp(-\lambda\,\|x-y\|)\|x-y\|^{-1}$ for a $\lambda \neq 0$ (fundamental solution of 3D Helmholtz equation),

where $K_0$ is the modified Bessel's function of the first kind and order 0. Note that all of them except the first one, are 3-times monotone radial kernels. In this paper also we will show that the fundamental solution of 2D Laplace equation can be extended to a 3-times monotone function if we get $\lambda = 1/(4\,\mathcal{D})$ when $\mathcal{D}$ is diameter of $\Omega$.

### 1.2. Literature review

There are some interesting results in the literature for the uniqueness of the solution of the BIE arising from Laplace equation. Researchers in [11–13], concluded that the BIE for Laplace equation with Dirichlet boundary conditions does not have a unique solution if the scaling of the domain is inappropriate. Now a question is how we can scale the boundary such that a unique solution can be obtained. Logarithmic capacity, $C_L$, is introduced in [14] for BIE with logarithmic kernels that is strongly related to the Euclidean diameter (see [15]) and it is shown that if $C_L = 1$ then there is no unique solution. Logarithmic capacity can be calculated explicitly for few domains, and its upper and lower bounds are reported in [16,17].

Authors of [18–22] have demonstrated that the condition number of the BEM-matrix for the Laplace equation in 2D with Dirichlet boundary conditions may become infinitely large when $C_L = 1$. Also it can be shown that the same result is valid for the equation with mixed boundary conditions [8]. Authors of [23–25] have shown that the condition number of the BEM-matrix for the Helmholtz equation may also become infinitely large. So in all these cases, the system is ill-conditioned and solving the linear system may be very difficult.

Authors of [26] studied the condition number of BEM-matrix for solving the Stokes equations on a two-dimensional domain supplemented with Dirichlet or mixed boundary conditions. They showed the condition number of the matrix is infinitely large, and for certain critical boundary contours, boundary integral equation is not solvable. Hence, for these critical contours the Stokes equations cannot be solved with the BEM. To overcome this problem, the connected domain can be rescaled.

In [27], ill-conditioning of BEM system is analytically examined by the use of degenerate kernel scheme. Five regularization techniques, namely hypersingular formulation, the method of adding a rigid body mode, rank promotion by adding the boundary flux equilibrium (direct BEM), the Combined Helmholtz Exterior Integral Equation Formulation (CHEIEF) method and the Fichera's method (indirect BEM), are analytically studied and numerically implemented to ensure the unique solution. The authors of [27] examined the sufficient and necessary conditions of boundary integral formulation

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