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Newton's algorithm for magnetohydrodynamic equations with the initial guess from Stokes-like problem

Sang Dong Kim [a], Eunjung Lee [b,*], Wonjoon Choi [b]

[a] Department of Mathematics, Kyungpook National University, Daegu 702-701, Republic of Korea

[b] Department of Computational Science and Engineering, Yonsei University, Seoul 120-749, Republic of Korea

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ABSTRACT

The magnetohydrodynamic equations are second order nonlinear partial differential equations which are coupled by fluid velocity and magnetic fields and we consider to apply the Newton's algorithm to solve them. It is well known that the choice of a proper initial guess is critical to assure the convergence of Newton's iterations in solving nonlinear partial differential equations. In this paper, we provide a good initial guess for Newton's algorithm when it is applied for solving magnetohydrodynamic equations.

1. Introduction

It is known that the magnetohydrodynamic equations have many applications in industry and engineering fields which may require understanding of plasmas, liquid metals, astrophysical flows and etc. (see [1,2] for example). Further, one may get a justification of using simplified magnetohydrodynamic equations to model magnetohydrodynamic flows in terrestrial [3] for example. A reduced stationary magnetohydrodynamic equations with magnetic and electric fields are studied by finite element approximation in [4,5]. Further, such stationary magnetohydrodynamics equations are extended to a time-dependent magnetohydrodynamics equations in [6], where two partitioned IMEX methods in finite element approximation are introduced to solve such an evolutionary magnetohydrodynamic equations. Some works on time-dependent magnetohydrodynamic may be found in [7,3,6,8–10].

The incompressible magnetohydrodynamic equations considered here, that are second order in the pair of fluid velocity and magnetic fields, describe the cosmic fluids which carry electric currents and their magnetic fields [11]. By omitting the displacement current, the magnetohydrodynamics equations can be simplified as

$$\begin{gathered}\rho\,(\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u})=-\nabla p-\lambda\mathbf{b}\times(\nabla\times\mathbf{b})+\nu\Delta\mathbf{u},\qquad \partial_t\mathbf{b}=\nabla\times(\mathbf{u}\times\mathbf{b})+\eta\Delta\mathbf{b},\\ \nabla\cdot\mathbf{u}=0,\qquad \nabla\cdot\mathbf{b}=0,\end{gathered}\tag{1.1}$$

where $\mathbf{u}$ and $\mathbf{b}$ are respective fluid velocity and magnetic fields; ρ is a constant mass density; $\lambda,\ \nu$ and η are conventional parameters which are circumstantially described in [11]. Obviously the magnetohydrodynamic equations are composed of nonlinear partial differential equations and solving this system requires a special treatment.

In this paper, without considering any time and space discretization to solve the targeting magnetohydrodynamic equations, we are focused on its linearization process. The Newton's algorithm is one of the most widely used linearization schemes to solve nonlinear partial differential equations. By using the first Fréchet derivative of a solution operator, it

* Corresponding author.
E-mail addresses: skim@knu.ac.kr (S.D. Kim), eunjunglee@yonsei.ac.kr (E. Lee), kgarrett0810@gmail.com (W. Choi).

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linearizes the process and constructs a sequence that is supposed to converge to the exact solution. It is well known that the initial guess is required to be chosen close enough to the exact solution in order to guarantee the quadratic convergence of Newton's iteration [12]. The goal of this paper is to provide a proper initial guess of Newton's algorithm for magnetohydrodynamic equations. We first recast the magnetohydrodynamic equations into familiar Navier–Stokes like equations. It has been studied in a frame work of finite element approximations that proper Stokes equations may be chosen to search for initial guess (see [13–15]) of the Newton's iteration for Navier–Stokes equations. Based on these studies, we then find the corresponding evolutionary Stokes-like equations and construct an approximate solution by interrelated Galerkin method using a passage to the limit and compactness theorem. The existence and uniqueness of the evolution Stokes-like equations are shown following the techniques in [12]. At last, we will show that the solution of the evolution Stokes-like problem is a proper initial guess to guarantee the convergence of Newton's iteration in solving the magnetohydrodynamic equations.

2. Magnetohydrodynamic equations in a vector form

The magnetohydrodynamic (MHD) equations with a constant mass density ρ can be described as

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p - \lambda \mathbf{b}\times(\nabla\times\mathbf{b}) + \nu\Delta\mathbf{u} \quad \text{in } Q := \Omega\times[0,T] \tag{2.1}$$

$$\partial_t \mathbf{b} = \nabla\times(\mathbf{u}\times\mathbf{b}) + \eta\Delta\mathbf{b} \quad \text{in } Q \tag{2.2}$$

$$\nabla\cdot\mathbf{u} = 0, \qquad \nabla\cdot\mathbf{b} = 0 \quad \text{in } Q \tag{2.3}$$

and $\mathbf{u}(\mathbf{x}, t_0) = \mathbf{u}_{\mathrm{IC}}$, $\mathbf{b}(\mathbf{x}, t_0) = \mathbf{b}_{\mathrm{IC}}$ in Ω, $\mathbf{u}(\Gamma, t) = \mathbf{u}_b$, $\mathbf{b}(\Gamma, t) = \mathbf{b}_b$ in $[0, T]$, where $\Gamma = \partial\Omega$, $\mathbf{u}$ and $\mathbf{b}$ are the velocity and magnetic fields respectively; p is the pressure which is divided by the constant mass density; ν and η are the kinematic viscosity and the magnetic resistivity, respectively, with $\lambda = \frac{1}{\mu\rho}$ and $\eta = \frac{1}{\mu\sigma}$ in which μ is a susceptibility and σ is an electric conductivity. Here, the spatial domain Ω is an open and bounded Lipschitz set in $\mathbb{R}^3$.

According to [11], by introducing new variables

$$\begin{bmatrix}\mathbf{z}^+\\ \mathbf{z}^-\end{bmatrix} = \mathbf{z}^\pm := \mathbf{u}\pm\lambda^{\frac{1}{2}}\mathbf{b} = \begin{bmatrix}\mathbf{u}+\lambda^{\frac{1}{2}}\mathbf{b}\\ \mathbf{u}-\lambda^{\frac{1}{2}}\mathbf{b}\end{bmatrix}, \qquad \begin{bmatrix}\nu^+\\ \nu^-\end{bmatrix} = \nu^\pm := \frac{1}{2}(\nu\pm\eta) = \frac{1}{2}\begin{bmatrix}\nu+\eta\\ \nu-\eta\end{bmatrix} \tag{2.4}$$

and

$$q = p + \frac{1}{8}|\mathbf{z}^+ - \mathbf{z}^-|^2, \tag{2.5}$$

the MHD equations in (2.1)–(2.3) can be expressed such that

$$\partial_t\mathbf{z}^+ + (\mathbf{z}^-\cdot\nabla)\mathbf{z}^+ + \nabla q - \Delta(\nu^+\mathbf{z}^+ + \nu^-\mathbf{z}^-) = \mathbf{0} \quad \text{in } Q := \Omega\times[0,T] \tag{2.6}$$

$$\partial_t\mathbf{z}^- + (\mathbf{z}^+\cdot\nabla)\mathbf{z}^- + \nabla q - \Delta(\nu^-\mathbf{z}^+ + \nu^+\mathbf{z}^-) = \mathbf{0} \quad \text{in } Q \tag{2.7}$$

$$\nabla\cdot\mathbf{z}^\pm = 0 \quad \text{in } Q \tag{2.8}$$

$$\mathbf{z}^\pm(\mathbf{x},0) = \mathbf{z}^\pm_{\mathrm{IC}}(\mathbf{x}) \quad \text{in } \Omega \tag{2.9}$$

$$\mathbf{z}^\pm = \mathbf{z}^\pm_{\mathrm{BC}} \quad \text{on } \Gamma\times[0,T]. \tag{2.10}$$

From now on, in all the sequel we assume that

$$\nu^- > 0, \quad \int_\Omega q\,d\Omega = 0 \quad \text{and} \quad \int_\Gamma \mathbf{z}^\pm\cdot\mathbf{n}\,ds = 0.$$

Let

$$\mathrm{M} = \begin{bmatrix}\nu^+ I_3 & \nu^- I_3\\ \nu^- I_3 & \nu^+ I_3\end{bmatrix}, \qquad \mathrm{P} = \begin{bmatrix}\mathbf{0} & I_3\\ I_3 & \mathbf{0}\end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix}q\\ q\end{bmatrix},$$

in which I_3 is the 3×3 identity matrix. For a $\mathbf{w} = [\mathbf{u},\mathbf{v}]^T$, we denote

$$\partial_t\mathbf{w} = \begin{bmatrix}\partial_t\mathbf{u}\\ \partial_t\mathbf{v}\end{bmatrix}, \qquad \Delta\mathbf{w} = \begin{bmatrix}\Delta\mathbf{u}\\ \Delta\mathbf{v}\end{bmatrix}, \qquad \mathbf{w}\cdot\nabla = \begin{bmatrix}\mathbf{u}\cdot\nabla\\ \mathbf{v}\cdot\nabla\end{bmatrix},$$

$$\nabla\mathbf{q} = \begin{bmatrix}\nabla q\\ \nabla q\end{bmatrix}, \qquad \mathrm{M}\mathbf{w} = \begin{bmatrix}\nu^+\mathbf{u}+\nu^-\mathbf{v}\\ \nu^-\mathbf{u}+\nu^+\mathbf{v}\end{bmatrix} \quad \text{and} \quad \mathrm{P}\mathbf{w} = \begin{bmatrix}\mathbf{v}\\ \mathbf{u}\end{bmatrix}.$$

With these notations and definitions, it should be understood as

$$\begin{bmatrix}(\mathbf{u}^-\cdot\nabla)\mathbf{u}^+\\ (\mathbf{u}^+\cdot\nabla)\mathbf{u}^-\end{bmatrix} = (\mathrm{P}\mathbf{u}^\pm\cdot\nabla)\mathbf{u}^\pm, \qquad \Delta(\mathrm{M}\mathbf{z}^\pm) = \begin{bmatrix}\Delta\nu^+\mathbf{z}^+ + \Delta\nu^-\mathbf{z}^-\\ \Delta\nu^-\mathbf{z}^+ + \Delta\nu^+\mathbf{z}^-\end{bmatrix} = \mathrm{M}\Delta\mathbf{z}^\pm. \tag{2.11}$$