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A parameter robust higher order numerical method for singularly perturbed two parameter problems with non-smooth data

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ABSTRACT

A singularly perturbed second order ordinary differential equation having two parameters with a discontinuous source term is presented for numerical analysis. Theoretical bounds on the derivatives, regular and singular components of the solution are derived. A hybrid monotone difference scheme with the method of averaging at the discontinuous point is constructed on Shishkin mesh. Parameter-uniform error bounds for the numerical approximation are established. Numerical results are presented which support the theoretical results.

1. Introduction

Singularly perturbed differential equations arise in many areas of applied mathematics and mathematical physics such as fluid dynamics, quantum mechanics, elasticity, chemical reactor theory, gas porous electrodes theory, meteorology, oceanography, rarefied gas dynamics, diffraction theory, reaction–diffusion process, non-equilibrium and radiating flows, Navier–Stokes equations of fluid flow at high Reynolds number, etc. The differential equation depends on a small positive parameter (ε), multiplying the highest derivative term. When the parameter tends to zero ($\varepsilon \to 0$) the problem has a limiting solution which is called the solution of the reduced problem [1] and the regions of non-uniform convergence lie near the boundary, which are known as boundary layers. These problems have steep gradients in the narrow layer regions of the domain in consideration. This causes severe hurdles in the computations for classical numerical methods. In order to capture the layers, a large number of special purpose methods have been developed by the researchers to provide accurate numerical solutions which cover second order equations with single parameter for smooth [1–3] and non smooth data [4–8]. In recent years, authors have considered singularly perturbed second order ordinary differential equation with two small parameters (ε, μ) in smooth data [9–12] and have considered non-smooth data [13,14]. These types of problems are widely found in many applications, for example the model transport phenomena in chemistry [15], Lubrication theory [16], Chemical reactor theory [17] and also in DC motor analysis [18].

In [19] Vigo Aguiar et al. considered a two point boundary value problem for second order ordinary differential equation. A boundary value technique is used in parallel computers to reduce the computation time and showed the reliability and

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performance of the proposed parallel schemes. In [20], the authors proposed a method for numerical solution of singularly perturbed two point boundary value problems, in which the second order BVP is converted into a system of IVPs and second order convergence is shown using exponentially fitted finite difference schemes. P. Das and V. Mehrmann discussed a singularly perturbed parabolic initial boundary value problem for 1-D convection–diffusion–reaction equation containing two small parameters. A moving mesh technique by the equidistribution of a positive monitor function is taken to generate meshes and it shows first order accuracy [21]. A system of coupled singularly perturbed reaction–diffusion problems having diffusion parameters with different magnitudes is considered in [22]. Central difference scheme is used to discretize the problem on equidistribution mesh to obtain an optimal second-order parameter uniform convergence.

Motivated by the works of [5,14,23], we have considered a singularly perturbed reaction–convection–diffusion equation in one dimension with a discontinuous source term of the form:

$$Ly(x) \equiv \varepsilon y''(x) + \mu a(x)y'(x) - b(x)y(x) = f(x), \quad x \in \Omega^- \cup \Omega^+, \tag{1}$$

$$y(0) = y_0, \quad y(1) = y_1 \tag{2}$$

$$|[f(d)]| \le C.$$

It is convenient to introduce the notations $\overline{\Omega} = [0, 1]$, $\Omega^- = (0, d)$ and $\Omega^+ = (d, 1)$, $a(x)$ and $b(x)$ are sufficiently smooth functions in $\overline{\Omega}$ and $f(x)$ is sufficiently smooth in $\Omega^- \cup \Omega^+ \cup \{0, 1\}$. Also $f(x)$ and its derivatives have a jump discontinuity at $d \in \Omega = (0, 1)$ (denoted by $[w](d) = w(d^+) - w(d^-)$), $0 < \varepsilon << 1$, $0 \le \mu \le 1$, $a(x) \ge \alpha > 0$, $b(x) \ge \beta > 0$ and $\rho = \min_{\overline{\Omega}} \left\{\frac{b}{a}\right\}$.

Under these assumptions, the SPP (1)–(2) has a solution $y(x) \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$, when $\mu = 1$ the problem is a well known convection–diffusion problem [7] and when $\mu = 0$, we get the reaction–diffusion problem [5,8]. In the present article the following cases are considered $\sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}$ and $\sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}$.

Throughout this article C denotes a generic positive constant independent of nodal points, mesh size (N) and the perturbation parameters ε, μ. We measure all functions in the supremum norm, denoted by

$$\|w\|_{\overline{\Omega}} = \sup_{x \in \overline{\Omega}} |w(x)|.$$

The structure of the paper is as follows. In Section 2, we establish an existence theorem for (1)–(2), minimum principle, stability result and some priori estimates on the solution and its derivatives. Section 3 presents a decomposition of the discrete solution to solve the problem, which generates robust numerical approximation to the solution. Truncation error analysis is estimated in Section 4. This analysis gears the main theoretical results presented in Section 5, ε–μ uniform convergence in the maximum norm of the approximations is generated by the numerical method. Numerical examples are provided in Section 6 to illustrate the applicability of the method with maximum pointwise errors, and rate of convergence in the form of tables.

2. A priori bounds on the solution and its derivatives

We commence this section by the following existence theorem.

Theorem 1. *The SPP (1)–(2) has a solution $y(x) \in C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$.*

Proof. The proof is by construction. Let $y_1(x)$, $y_2(x)$ be particular solutions of the differential equations

$$\varepsilon y_1''(x) + \mu a(x)y_1'(x) - b(x)y_1(x) = f(x), \quad x \in \Omega^- \quad \text{and}$$
$$\varepsilon y_2''(x) + \mu a(x)y_2'(x) - b(x)y_2(x) = f(x), \quad x \in \Omega^+.$$

Consider the function

$$y(x) = \begin{cases} y_1(x) + (y(0) - y_1(0))\phi_1(x) + A\phi_2(x), & x \in \Omega^- \\ y_2(x) + B\phi_1(x) + (y(1) - y_2(1))\phi_2(x), & x \in \Omega^+ \end{cases}$$

where $\phi_1(x)$, $\phi_2(x)$ are the solutions of the boundary value problems

$$\varepsilon\phi_1''(x) + \mu a(x)\phi_1'(x) - b(x)\phi_1(x) = 0, \quad x \in \Omega, \; \phi_1(0) = 1, \; \phi_1(1) = 0$$
$$\varepsilon\phi_2''(x) + \mu a(x)\phi_2'(x) - b(x)\phi_2(x) = 0, \quad x \in \Omega, \; \phi_2(0) = 0, \; \phi_2(1) = 1$$

and A, B are constants to be chosen so that $y(x) \in C^1(\Omega)$.

Note that on the open interval $(0, 1)$, $0 < \phi_i < 1$, $i = 1, 2$. Thus ϕ_1, ϕ_2 cannot have an internal maximum or minimum and hence

$$\phi_1'(x) < 0, \qquad \phi_2'(x) > 0, \quad x \in (0, 1).$$

We wish to choose the constants A, B so that, $y(x) \in C^1(\Omega)$. That is, we impose

$$y(d-) = y(d+) \quad \text{and} \quad y'(d-) = y'(d+).$$