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A combined discontinuous Galerkin finite element method for miscible displacement problem

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A new combined method is constructed for solving incompressible miscible displacement in porous media. In this procedure, a hybrid mixed element method is constructed for the pressure and velocity equation, while a symmetric discontinuous Galerkin finite element method is proposed for the concentration equation. The introduction of these two numerical methods not only makes the coefficient matrixes symmetric positive definite, but also conserves the local mass balance. The stability and consistency of the method are analyzed and the optimal error estimate in $L^\infty(L^2)$ for velocity and concentration and the super convergence in $L^\infty(H^1)$ for pressure are derived. Finally, some numerical results are presented.

1. Introduction

Numerical simulation of miscible displacement in porous media is very important and interesting in oil reservoir and environmental pollution. It has attracted considerable attention during recent decades and becomes an important field in modern computational mathematics. In this paper, we consider the following two-phase (water and oil) miscible displacement problem in porous media, which is governed by a nonlinear coupled system of partial differential equations [1–5]:

$$\begin{aligned} &(\text{a})\ \nabla\cdot\mathbf{u}=q, \qquad \mathbf{u}=-\frac{k^*}{\mu(c)}\nabla p, && (x,t)\in\Omega\times(0,T],\\ &(\text{b})\ \phi\frac{\partial c}{\partial t}+\mathbf{u}\cdot\nabla c-\nabla\cdot(D(\mathbf{u})\nabla c)=q(c^*-c), && (x,t)\in\Omega\times(0,T], \end{aligned} \tag{1.1}$$

with the following initial–boundary conditions:

$$\begin{aligned} \mathbf{u}\cdot\nu &= 0, \qquad (x,t)\in\partial\Omega\times(0,T],\\ D(\mathbf{u})\nabla c\cdot\nu &= 0, \qquad (x,t)\in\partial\Omega\times(0,T],\\ c(x,0) &= c^0(x),\ \ x\in\Omega \end{aligned} \tag{1.2}$$

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where Ω is a bounded polygonal/polyhedral domain in R^d $(d = 2,\ 3)$ and c denotes the concentration of the fluid mixture; q is the external volumetric flow rate; c^* is the concentration in the external flow, which must be specified at injection points $(q > 0)$ and is assumed to be equal to c at production points $(q < 0)$; $k^* = k^*(x)$ and $\phi = \phi(x)$ are the permeability and the porosity of the rock, respectively. $\mu(c)$ is the viscosity of the fluid; p is the pressure of fluid and $\mathbf{u}$ is Darcy velocity. $D(\mathbf{u}) = \phi[d_m I + |\mathbf{u}|(d_l E(\mathbf{u}) + d_t E^{\perp}(\mathbf{u}))]$ denotes the diffusion matrix describing the effects of molecular diffusion and dispersion, where $E(\mathbf{u}) = (u_i u_j/|\mathbf{u}|^2)_{d\times d}$ is the $d \times d$ matrix representing orthogonal projection along the velocity vector, and $E^{\perp}(\mathbf{u}) = I - E(\mathbf{u})$ is its orthogonal complement, d_m, d_l, and d_t are the molecular diffusion, longitudinal and transverse dispersion coefficients, respectively. ν is the unit outer normal vector to boundary $\partial\Omega$. Furthermore, a compatibility condition $\int_\Omega q(x,t)dx = 0$ must be imposed to determine the pressure.

In the paper, we consider a new combined method for the above nonlinear coupled equations. Firstly, we consider a numerical method for the velocity equation. More accurate approximation of the velocity function can be achieved by traditional mixed element methods [1–3,6]. However, the technique of the traditional mixed element methods leads to saddle point problems, where the finite element spaces require the LBB condition and the coefficient matrix of the mixed system loses the symmetric property. Hybrid mixed element methods [7–10] are characterized by the removal of the continuity of the normal component of the velocity over each element interface, resulting in a symmetric positive definite system. In this paper, we consider the hybrid mixed element method to solve velocity and pressure equations, in which, the subproblems for velocity and pressure are solved at element level and these variables are eliminated in favor of the Lagrange multiplier, identified as pressure trace at the element interfaces, and the global system involves only the degrees of freedom associated with the multiplier, significantly reducing the computational cost.

Then, we turn to the numerical scheme for the concentration equation. The concentration equation is parabolic and normally convection-dominated. The standard Galerkin method applied to the convection-dominated problems does not work well, especially for the discontinuous problem. Discontinuous Galerkin finite element (DGFE) methods were introduced independently in [11–13]. Since then, they have become more and more powerful tools to deal with many types of discontinuous problems [14–17]. DGFE methods have several advantages over other types of finite element methods. For example, no continuity constraints are explicitly imposed on the trial and test functions across the finite element interfaces, thus the spaces are easy to construct, and the use of highly nonuniform and unstructured meshes is permitted. A combined classical mixed element scheme with DGFE method for incompressible miscible displacement problems was presented in [16,17]. The similar method was extended in [18] to solve the compressible miscible displacement problem. The DGFE techniques have been applied by the authors of this paper [19–22], to nonlinear partial differential equations.

The main purpose of this paper is to combine the hybrid mixed element method with discontinuous Galerkin finite element method to solve incompressible miscible displacement problem in porous media. Compared with other combined methods, this method allows to significantly reduce the number of the globally coupled degrees of freedom and deal with the discontinuous problem well. The stability and consistency of the method are analyzed and the optimal error estimates in $L^\infty(L^2)$ for velocity and concentration and the super convergence in $L^\infty(H^1)$ for pressure are derived. Finally, some numerical results are presented to confirm the theoretical analysis.

2. Hybrid mixed discontinuous Galerkin finite element method

2.1. Basic assumptions and notation

In this paper, we adopt notations and norms of usual Sobolev spaces. For convenience, we make the following assumptions on the coefficients in (1.1): Let a_*, a^*, ϕ_*, ϕ^* and K be some positive constants such that

$$\begin{aligned} &0 < \phi_* \le \phi(x) \le \phi^*, \qquad 0 < a_* \le \frac{k^*(x)}{\mu(c)} \le a^*, \\ &\left|\frac{\partial}{\partial c}\left(\frac{k^*(x)}{\mu(c)}\right)\right| + |q(x,t)| \le K. \end{aligned} \tag{2.3}$$

Let $\{\mathcal{T}_h\}$ be a quasi-uniform regular partition of Ω with $\mathcal{T}_h = \{T_1, T_2, \ldots, T_{N_h}\}$, the element T_i is triangle or quadrilateral $(d = 2)$, or tetrahedron $(d = 3)$. The partition is conforming as both the pressure/velocity and transport equations will be used the same mesh. Denote $\partial\mathcal{T}_0$ the set of the common faces of $T_i \cap T_j$ for all neighboring $T_i, T_j \in \mathcal{T}_h$. Correspondingly, let $\partial\mathcal{T}_1$ denote the set of the common faces of $T \cap \partial\Omega$ for all $T \in \mathcal{T}_h$ which has at least two points in 2D and at least three points in 3D on the $\partial\Omega$. Set $\partial\mathcal{T}_h = \partial\mathcal{T}_0 \cup \partial\mathcal{T}_1$.

Since discontinuous finite element is used, we introduce the piecewise Sobolev spaces given by

$$\begin{aligned} &\mathcal{H}^s(\mathcal{T}_h) = \left\{z \in L^2(\Omega) : z|_T \in H^s(T),\ \forall\, T \in \mathcal{T}_h\right\}, \quad s \ge 0, \\ &L^2(\partial\mathcal{T}_h) = \{\mu \in L^2(e), \forall\, e \in \partial\mathcal{T}_h\}, \\ &L^2(\partial\mathcal{T}_0) = \{\mu \in L^2(e), \forall\, e \in \partial\mathcal{T}_0\}. \end{aligned}$$