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Reconstruction of the electric field of the Helmholtz equation in three dimensions

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ABSTRACT

In this paper, we rigorously investigate the truncation method for the Cauchy problem of Helmholtz equations which is widely used to model propagation phenomena in physical applications. The truncation method is a well-known approach to the regularization of several types of ill-posed problems, including the model postulated by Regińska and Regiński (2006). Under certain specific assumptions, we examine the ill-posedness of the non-homogeneous problem by exploring the representation of solutions based on Fourier mode. Then the so-called regularized solution is established with respect to a frequency bounded by an appropriate regularization parameter. Furthermore, we provide a short analysis of the nonlinear forcing term. The main results show the stability as well as the strong convergence confirmed by the error estimates in L^2-norm of such regularized solutions. Besides, the regularization parameters are formulated properly. Finally, some illustrative examples are provided to corroborate our qualitative analysis.

1. Introduction

The scalar Helmholtz equation is the well-spring of many streams in both mathematical and engineering problems due to the formal equivalence of the wave equation (and the Schrödinger equation for further applications). To set the stage for our problem presented in this paper, we review very briefly the relation between these equations using simple computations and elementary techniques. In fact, the scalar wave equation that derives the Helmholtz equation is simply expressed by

$$v_{tt} = c^2 \Delta v + F(t,x), \tag{1.1}$$

where c denotes the local speed of propagation for waves, and $F(t,x)$ is a source that injects waves into the solution. Suppose that we look for a solution with the wave number $k = 1/\lambda > 0$ defined by wavelength λ, and that the source generates waves of this type, i.e.

$$v(t,x) = u(x)\, e^{-ikt}, \qquad F(t,x) = q(x)\, e^{-ikt}.$$

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Substituting these quantities into (1.1), then dividing by e^{-ikt} and reordering the terms, we obtain

$$\Delta u(x) + \frac{k^2}{c^2} u(x) = -\frac{q(x)}{c^2}.$$

This is the three-dimensional non-homogeneous Helmholtz equation in which we are interested here, which mathematically reads

$$\Delta u(x) + k^2 n u(x) = -f(x), \tag{1.2}$$

where the coefficient $n = 1/c^2$ (normalized in this paper) is in principle known as the index of refraction [1,2]; and $f(x) = q(x)/c^2$ represents the forcing term.

In this paper, we continue the work that commenced in [3] by Regińska and Regiński, where the Cauchy problem of Helmholtz equations in particular drives us to the model of reconstruction of the whole radiation field in optoelectronics. This problem is associated with Hadamard-instability due to the fact that the high frequency modes grow exponentially fast. Typically, it would imply the severe ill-posedness as well as the impossibility of solving the problem. Hence, it is customary to overcome this difficulty via a regularization method.

In the context of regularization methods, the homogeneous problem ($f \equiv 0$) has been studied mathematically for more than a decade. Recent developments of theoretical computations have been achieved, e.g. the truncation method in [3], the quasi-reversibility-type method in [4], and the Tikhonov-type method in [5,6], where the energy of solution is supposedly known in some certain cases. On the other side, various boundary element regularization methods are solidly compared in [7]. We, nevertheless, stress that the qualitative analysis of instability from the above-mentioned works mostly lacks theoretical validation. Even though the authors in [3] rigorously investigated the discernible impact of physical parameters upon independence of solution on given data, a convincing example seems to be needed.

Currently, there have been many other fields of study where the Helmholtz-type equations can be greatly used, such as the influence of the frequency on the stability of Cauchy problems [8], finding the shape of a part of a boundary in [9], regularization of the modified Helmholtz equation in [10] and the problem of identifying source functions in [11,12]. As we can see from the references, while the literature on the homogeneous problem is very extensive, there would have to emerge some potential field to consider the non-homogeneous problem (and the nonlinear case which we shall figure out later on). Even though the homogeneous problem has been solved massively, it immediately raises a question: Is it possible to use those methods when the forcing term dominates? It surely requires highly sophisticated techniques and all surrounding issues need to be invented due to the occurrence of new parts. To the best of our knowledge, rigorous investigation of the instability regime and qualitative analysis of the truncation approach have so far not been considered for the Helmholtz equation with genuinely mixed boundary conditions. Moreover, both the Helmholtz equation and the truncation method are ubiquitous in applied mathematics. It is thus imperative to answer the above question with full details.

Summarizing, our main objectives are

- proving the underlying model is unstable in the sense of Hadamard and giving a theoretical example for such instability;
- applying the truncation method to define the regularized solution and showing the error estimates which also imply the stability and strong convergence;
- providing a short extension of the problem with a nonlinear forcing term.

The remaining part of this paper is organized as follows. In Section 2, we state the model problem, introduce the abstract settings, and herein discuss thoroughly the nature of ill-posedness. Section 3 is devoted to our second objective whilst the third objective is investigated in Section 4. As a result, the error estimates together with stability are proved with respect to measurement level and we present explicit formulae for the regularization parameters. Our analysis is mainly based on the Fourier transform, superposition principle and Parseval's identity. Interestingly, we observe that when choosing a suitable regularization parameter, the convergence rate stays unchanged from the linear case to the nonlinear case. Numerical tests are provided in Section 5 to illustrate our method and Section 6 concludes the paper with a discussion of our results and forthcoming aims.

2. Abstract settings and Ill-posedness

2.1. Abstract settings

Let us consider the problem of reconstructing the radiation field $u = u(x, y, z)$ in the domain $\Omega = \mathbb{R}^2 \times (0, d)$, $d > 0$. For simplicity, the first two variables will be denoted by $\xi := (x, y)$. The problem given by (1.2) along with the boundary conditions can be written as follows:

$$\begin{cases} \Delta u + k^2 u = -f, & \text{in } \Omega, \\ u(\xi, d) = g(\xi), & \xi \in \mathbb{R}^2, \\ \partial_z u(\xi, d) = h(\xi), & \xi \in \mathbb{R}^2, \\ u(\cdot, z) \in L^2(\mathbb{R}^2), & z \in [0, d], \end{cases} \tag{2.1}$$

where $g, h \in L^2(\mathbb{R}^2)$ are given data and $f \in L^2(\Omega)$ plays a role as the given forcing term.