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Nonlinear Darcy fluid flow with deposition

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ABSTRACT

In this article we consider a model of a filtration process. The process is modeled using the nonlinear Darcy fluid flow equations with a varying permeability, coupled with a deposition equation. Existence and uniqueness of the solution to the modeling equations is established. A numerical approximation scheme is proposed and analyzed, with an a priori error estimate derived. Numerical experiments are presented which support the obtained theoretical results.

1. Introduction

Filtration processes are ubiquitous in our lives. From oil and fuel filters in engines, to filters used in industrial lines to protect sensitive equipment, to household water filtration systems. In this article we consider the case where the filtration process can be modeled as a fluid flowing through a porous medium. We make the simplifying assumption that the rate of particulate deposition in the filter is only dependent on the porosity and the magnitude of the fluid velocity at that point. Of interest are the modeling equations

$$\frac{\mu_{eff}}{\kappa(\eta)}\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \tag{1.2}$$

$$\frac{\partial \eta}{\partial t} + dep(|\mathbf{u}|, \eta) = 0, \quad \text{in } \Omega\,, \tag{1.3}$$

subject to suitable boundary and initial conditions. In (1.1)–(1.3) $\mathbf{u}$ and p denote the velocity and pressure of the fluid, respectively, μ_{eff} the effective fluid viscosity, and η and $\kappa(\eta)$ represent the porosity and permeability throughout the filter (Ω), respectively. (A discussion of the model follows in Section 2.)

The lack of regularity of the fluid velocity, $\mathbf{u} \in H_{div}(\Omega)$, leads to an open question of the existence of a solution to (1.1)–(1.3). In order to obtain a modeling system of equations for which a solution can be shown to exist, we replace η in (1.1) by an *smoothed* porosity, η^s. The approach of regularizing the model with the introduction of η^s is, in part, motivated by the Darcy fluid flow equations, which can be derived by *averaging*, e.g. volume averaging [1], homogenization [2], mixture theory [3]. Recently in [4] we considered the case of (steady-state) generalized Newtonian fluid flow through a porous medium, modeled by Eqs. (1.1), (1.2), with $\frac{\mu_{eff}}{\kappa(\eta)} \longrightarrow \beta(|\mathbf{u}|)$. With the general assumptions that $\beta(\cdot)$ was a positive, bounded, Lipschitz continuous function, bounded away from zero, and with $\beta(|\mathbf{u}|)$ replaced with $\beta(|\mathbf{u}^s|)$, existence of a solution was established. Two smoothing operators for $\mathbf{u}$ were presented. One was a local averaging operator, whereby $\mathbf{u}^s(\mathbf{x})$ is obtained

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by averaging $\mathbf{u}$ in a neighborhood of $\mathbf{x}$. The second smoothing operator, which is nonlocal, computes $\mathbf{u}^s(\mathbf{x})$ using a differential filter applied to $\mathbf{u}$. That is, $\mathbf{u}^s$ is given by the solution to an elliptic differential equation whose right hand side is $\mathbf{u}$. For establishing the existence of a solution the key property that the smoothing operator needs to satisfy is that it transforms a weakly convergent sequence in $L^2(\Omega)$ into a sequence which converges strongly in $L^\infty(\Omega)$.

A similar model to (1.1)–(1.3) arises in the study of single-phase, miscible displacement of one fluid by another in a porous medium. For this problem η would denote a fluid concentration, and the hyperbolic deposition equation (1.3) is replaced by a parabolic transport equation. Existence and uniqueness for this problem has been investigated and established by Feng [5] and Chen and Ewing [6]. Because of the connection of this model to oil extraction, numerical approximation schemes for this problem have been well established. A summary of these methods is discussed in the recent papers by Bartels, Jensen and Müller [7], and Riviére and Walkington [8].

A steady-state nonlinear Darcy fluid flow problem, with a permeability dependent on the pressure was investigated by Azaïez, Ben Belgacem, Bernardi, and Chorfi [9], and Girault, Murat, and Salgado [10]. For the permeability function Lipschitz continuous, and bounded above and below, existence of a solution $(\mathbf{u}, p) \in L^2(\Omega) \times H^1(\Omega)$ was established. Important in handling the nonlinear permeability function, in establishing existence of a solution, was the property that $p \in H^1(\Omega)$. In [9] the authors also investigated a spectral numerical approximation scheme for the nonlinear Darcy problem, assuming an axisymmetric domain Ω. A convergence analysis for the finite element discretization of this problem was given in [10].

Following a discussion of the model in Section 2, existence of a solution to the modeling equations is established in Section 3. An approximation scheme for the filtration model is presented in Section 4, and an a priori error estimate derived. A numerical simulation of the filtration process is presented in Section 5.

2. Discussion of filtration model

In this section we discuss the modeling equations we investigate for the filtration process. We assume that the process can be modeled as fluid flowing through a porous medium with a varying permeability. Additionally we assume that the process has a fixed time horizon, T. (For example, for industrial filters the most practical time to change filters is during scheduled maintenance periods. Drivers are encouraged to change the oil filters in their cars every 3000 miles or every three months, whichever comes first.) We use the following parameters/variables to model the process.

$\Omega \subset \mathbb{R}^d (d = 2, 3)$ — the region occupied by the filter,
$\mathbf{u}$ — the fluid velocity,
p — the fluid pressure,
η — the porosity of the medium, $0 < \eta < 1$,
κ — the permeability of the medium, $0 < \kappa < \infty$,
μ_{eff} — the effective fluid viscosity,
$\mathbf{n}$ — the unit outer normal to Ω.

We model the fluid flow using the Darcy fluid flow equations:

$$\frac{\mu_{eff}}{\kappa(\eta)}\mathbf{u} + \nabla p = \mathbf{0}, \quad \text{in } \Omega,\ t \in (0, T], \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega,\ t \in (0, T]. \tag{2.2}$$

Note: We are assuming that the particulate is sufficiently sparsely distributed in the fluid that the conservation of mass equation (2.2) is still a valid approximation for the model.

Relationship between permeability κ and porosity η

As $\eta \to 0$ the "porous" medium transitions to a "solid" medium, in which case the permeability, $\kappa \to 0$. As $\eta \to 1$ the medium's resistance to the flow goes to zero, i.e., its permeability goes to infinity, and the modeling equations are no longer appropriate to describe the fluid flow.

There are a number of postulated models for the relationship between κ and η. The most commonly cited is the Blake–Kozeny model [11]

$$\kappa(\eta) = \frac{D_p^2\, \eta^3}{150\,(1-\eta)^2}, \tag{2.3}$$

where D_p represents a material constant — the diameter of the particles comprising the porous medium.

Remark. The permeability of granite is $\approx 10^{-3} - 10^{-4}$ millidarcy. In a filtering process the permeability throughout Ω will always be greater than that of granite. So, it is reasonable to assume that $\kappa(\eta)$ is bounded from below. At the beginning of the filtering process there is a prescribed permeability (porosity) throughout Ω. As the filtering process decreases the permeability (porosity) throughout Ω it is also reasonable to assume that $\kappa(\eta)$ is bounded from above.