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Quasi-Newton minimization for the $p(x)$-Laplacian problem

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ABSTRACT

We propose a quasi-Newton minimization approach for the solution of the $p(x)$-Laplacian elliptic problem, $x \in \Omega \subset \mathbb{R}^m$. This method outperforms those existing for the $p(x)$-variable case, which are based on general purpose minimizers such as BFGS. Moreover, when compared to *ad hoc* techniques available in literature for the p-constant case, and usually referred to as "mesh independent", the present method turns out to be generally superior thanks to better descent directions given by the quadratic model.

1. Introduction

We consider the $p(x)$-Laplacian elliptic problem

$$\begin{cases} -\mathrm{div}(|\nabla u(x)|^{p(x)-2}\,\nabla u(x)) = f(x) & x \in \Omega \subset \mathbb{R}^m, \\ u(x) = 0 & x \in \partial\Omega \end{cases} \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^m$ with $\partial\Omega$ Lipschitz continuous, $p \in \mathcal{P}^{\log}$, that is p is a measurable function, $p\colon \Omega \to [1,+\infty]$ and $1/p$ is globally log-Hölder continuous. Moreover, we assume $1 < p_{\min} \le p(x) \le p_{\max} < \infty$, $f \in L^{p'(x)}(\Omega)$ (where $p'(x)$ denotes the dual variable exponent of $p(x)$) and $u \in V = W_0^{1,p(x)}(\Omega)$. Since $p(x)$ is bounded, we may see the space $W_0^{1,p(x)}(\Omega)$ as the space of functions in $W^{1,p(x)}(\Omega)$ with null trace on $\partial\Omega$. The trace operator can be defined on $W^{1,p(x)}(\Omega)$ in such a way that, as usual, if $u \in W^{1,p(x)}(\Omega) \cap \mathcal{C}(\overline{\Omega})$, then its trace coincides with $u|_{\partial\Omega}$. We refer to [1] for a general introduction to variable exponent Sobolev spaces. This model occurs in many applications, such as image processing [2,3] and electrorheological fluids [4–6], in which $p(x)$ may assume values close to the extreme ones [7–9]. Hereafter we leave the explicit dependence on $x \in \Omega \subset \mathbb{R}^m$ only for the exponent $p(x)$ and all integrals are intended over the domain Ω. The $p(x)$-Laplacian problem (1) admits a unique [10] weak solution $\underline{u}$ satisfying

$$\underline{u} = \arg\min_{v \in V} J(v)$$

where

$$J(u) = \int \frac{|\nabla u|^{p(x)}}{p(x)} - \int fu \tag{2}$$

or, equivalently,

$$J'(\underline{u})v = 0, \quad \forall v \in V \tag{3}$$

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where

$$J'(\underline{u})v = \int |\nabla \underline{u}|^{p(x)-2}\nabla \underline{u} \cdot \nabla v - \int f v. \tag{4}$$

A common way [11–14] to tackle the problem is the direct minimization, in a suitable finite dimensional subspace of V, of the functional J in Eq. (2), rather than solving the nonlinear equation (3) [15]. However, to our knowledge, *ad hoc* minimization algorithms were developed only for the p-constant case [13–15], whereas only general purpose methods such as the quasi-Newton method BFGS (Broyden–Fletcher–Goldfarb–Shanno) have been used for the $p(x)$-variable case [12].

In this work we minimize $J(u)$ employing a new quadratic model which makes use of the exact second differential $J''(u)$, only slightly regularized in order to handle possible analytic or numerical degeneracy when $|\nabla u|$ is small and $p(x)$ is close to the extreme values $p_{\min}$ or $p_{\max}$. The result is an efficient and robust algorithm converging faster than those available in literature, both for the p-constant case and the $p(x)$-variable one.

2. Minimization problem

We minimize $J(u)$ in a suitable finite element subspace of V and we call $\underline{u}^h$ the solution

$$\underline{u}^h = \arg \min_{v^h \in V_0^h} J(v^h) \Leftrightarrow J'(\underline{u}^h)v^h = 0 \quad \forall v^h \in V_0^h.$$

Given a regular triangulation of a polygonal approximation Ω_h of the domain, we select the subspace $V_0^h \subset V$ of continuous piecewise linear functions which are zero at the boundaries of Ω_h. Since for $p \neq 2$ problem (1) is degenerate quasi-linear elliptic, its solution has a limited regularity (see, for instance, [16]) and therefore higher-order finite element approximations do not worth (see Ref. [17]). For the variable exponent case, $p(x)$ is approximated by continuous piecewise linear functions as well, even if a local approximation by constant functions is possible (see Ref. [10,18]). Given the approximation $u^n \in V_0^h$ of the solution $\underline{u}^h$ at iteration n, we look for a direction $d^n \in V_0^h$ such that

$$J(u^n + \alpha_n d^n) < J(u^n).$$

The descent direction d^n is called *steepest descent* direction if

$$J'(u^n)d^n = -\left\|J'(u^n)\right\|_* \left\|d^n\right\|$$

where $\|\cdot\|$ is a suitable norm in V_0^h and $\|\cdot\|_*$ its dual norm. The idea (see Ref. [13,14]) is to find d^n as the solution of

$$d^n : b_n(d^n, v) = -J'(u^n)v, \quad \forall v \in V_0^h$$

where $b_n(\cdot,\cdot)$ is a suitable bilinear form depending on iteration n. The choice of b_n characterizes the minimization method.

The extension to non-homogeneous Dirichlet boundary conditions is straightforward. The solution $\underline{u}$ belongs to the variable exponent Sobolev space $W_g^{1,p(x)} = \{v \in W^{1,p(x)} : v = g \text{ on } \partial\Omega\}$ and its piecewise approximation must be in the space $V_{g_h}^h$, that is the space of continuous piecewise linear functions whose value of $\partial\Omega_h$ is g_h, where g_h is chosen to approximate the Dirichlet boundary data. The search directions are still in the space V_0^h.

2.1. Gradient-based directions

The choice in Ref. [13], for the p-constant case, is $d^n = w^n$, where

$$b_n(w^n, v) = \begin{cases} \int (\varepsilon + \left|\nabla u^n\right|^{p-2})\nabla w^n \cdot \nabla v, & p > 2 \\ \int (\varepsilon + \left|\nabla u^n\right|)^{p-2}\nabla w^n \cdot \nabla v, & p < 2. \end{cases} \tag{5}$$

The bilinear form $b_n(\cdot,\cdot)$ corresponds to a simple linearization of $J'(u^n)v$. The parameter ε is introduced in order to handle possible analytic or numerical degeneracy where $|\nabla u^n|$ is small. In fact, for $p \gg 2$ the term $|\nabla u^n|^{p-2}$ may underflow even if $|\nabla u^n| > 0$. On the other hand, for $p < 2$ the same term may overflow. We notice that the parameter ε is introduced only for finding the descent direction and not for regularizing the original $p(x)$-Laplacian functional J. With the above choice, the authors in Ref. [13] proved a convergence result ($J(u^n) \to J(u)$) only for the case $p > 2$. Their complicated proof is hardly extendible to the case $p < 2$ or to the general case with variable $p(x)$. The direction w^n is called in Ref. [13] *preconditioned steepest descent*. The scalar value α_n is chosen by exact line search

$$\alpha_n = \arg \min_{\alpha} J(u^n + \alpha d^n). \tag{6}$$

In Ref. [14] w^n is computed for all $1 < p < +\infty$ using the first definition in (5). The descent direction is then computed by

$$d^n = w^n + \beta_n d^{n-1}$$