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Uniform ultimate boundedness of numerical solutions to nonlinear neutral delay differential equations[☆]

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ABSTRACT

This paper is concerned with the long-time behaviour of the numerical solutions generated by Runge–Kutta (RK) methods for nonlinear neutral delay differential equations (NDDEs). It is proved that the numerical solutions produced by (k, l)-algebraically stable RK methods are uniformly ultimately bounded. Some examples reveal that some RK methods completely preserve the long-time behaviour of the exact solutions to NDDEs for sufficiently small time stepsize h. As a comparison with the previous results, a numerical example which further illustrates our theoretical results is provided.

1. Introduction

In this paper we are interested in the long-time behaviour of the numerical solution, which is obtained by a Runge–Kutta (RK) method with fixed time stepsize, to nonlinear neutral delay differential equations

$$y'(t) = f(t, y(t), y(t-\tau), y'(t-\tau)), \quad t \geq 0, \tag{1.1}$$

$$y(t) = \phi(t), \quad \tau \leq t \leq 0, \tag{1.2}$$

where τ is a real constant, the initial function $\phi \in C^1[-\tau, 0]$ with $C^q[-\tau, 0]$ denoting a set consisting of all q times continuously differentiable mapping $x : [-\tau, 0] \to \mathbf{X}$ for any given integer $q \geq 0$, on which the norm is defined by

$$x \in C^q[-\tau, 0], \qquad \|x\|_{C^q[-\tau,0]} = \sum_{i=0}^{q} \max_{t\in[-\tau,0]} \|x^{(i)}(t)\|.$$

Especially, $C^0[-\tau, 0]$ will be simply denoted by $C[-\tau, 0]$. Here $\mathbf{X}$ is a dense continuously embedded subspace of $\mathbf{H}$ which is a real or complex Hilbert space with inner product $\langle\cdot, \cdot\rangle$ and corresponding norm $\|\cdot\|$. Eq. (1.1) includes as an important special case in the non-neutral delay differential equations

$$y'(t) = f(t, y(t), y(t-\tau)), \quad t \geq 0. \tag{1.3}$$

Neutral delay differential equations (NDDEs) and their special cases have found applications in many areas of science (see, e.g., [1–3]). There has been significant interest in NDDEs recently, and a multitude of papers has been focussed on the linear stability of numerical method (see, e.g., [4–11]). The contractivity and asymptotic stability of the exact and numerical

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solutions to nonlinear NDDEs (1.1)–(1.2) have been studied by some authors (see [12–18]). These classical theories as laid out in Bellen and Zennaro [4] are devoted to the numerical stability analysis of initial value problems with simple dynamics.

Recently there have been a number of studies of numerical stability for wider classes of problems admitting more complicated dynamics. The uniform ultimate boundedness, which is defined precisely in Section 2, of the exact solutions to nonlinear NDDEs (1.1) has been proved under some conditions in [19]. In the context of their analysis, from a numerical point of view, it is important to study the potential of numerical methods in preserving the qualitative behaviour of the exact solutions. Thus in this paper we studied the uniform ultimate boundedness of numerical solutions defined by a RK method for nonlinear NDDEs (1.1).

The uniform ultimate boundedness is referred to as dissipativity in the research field of ODEs dynamical systems. The dissipative systems, which are characterized by a bounded set into which every orbit eventually enters and remains, generated by ODEs and their numerical counterpart have been investigated (see, e.g. [20–24]). The uniform ultimate boundedness of the exact solutions and the numerical solutions to non-neutral DDEs (1.3) has been reported in [25,26]. Later on, the long-time behaviour of numerical methods for DDEs with a bounded variable delay and functional differential equations (FDEs) was further investigated (see, e.g., [27–30]). It is worthy to note that Wang and Li [31] and Wang and Zhang [32] have studied the analytical and numerical dissipativity of nonlinear NDDEs with piecewise constant delay and with proportional delay, respectively; Gan [33] has studied the uniform ultimate boundedness of the exact solutions and the numerical solutions produced by θ-methods to nonlinear NDDEs (1.1) with the mapping f having special form

$$y'(t) = f(t, y(t), G(t, y(t-\tau), y'(t-\tau))), \quad t \geq 0. \tag{1.4}$$

Now in [19] having shown the uniform ultimate boundedness of the exact solutions to nonlinear NDDEs (1.1), we would like to show the uniform ultimate boundedness of the numerical solutions defined by a RK method.

In Section 2, we first collect several definitions and results from the literature, and discuss the numerical approximations to (1.1)–(1.2) by RK methods. Then the uniform ultimate boundedness result of (k, l)-algebraically stable RK methods for the system (1.1)–(1.2), where f satisfies the conditions assumed in Section 2, is established in Section 3. Some examples given in this section reveal that some RK methods completely preserve the long-time behaviour of the exact solutions to NDDEs (1.1) for sufficiently small time stepsize h. In Section 4, some numerical experiments are shown to illustrate the theoretical results exposed in this paper.

2. Preliminaries

Before stating the main results of the work, we first review some basic definitions and other essential mathematical preliminaries setting notations that will be used below.

2.1. Contractivity, asymptotic stability and uniform ultimate boundedness

As mentioned in Introduction, the classical theory of stability is devoted to the contractivity and asymptotic stability of solutions to differential equations, which can be stated as follows (see, for example, [21,22,4]).

Definition 2.1. Let $y(t)$ and $z(t)$ be the solutions of (1.1) corresponding to the initial functions $\phi(t)$ and $\varphi(t)$, respectively. We say that the system (1.1) is contractive if $y(t)$ and $z(t)$ satisfy

$$\|y(t) - z(t)\| \leq \max_{-\tau \leq s \leq 0} \|\phi(s) - \varphi(s)\|, \quad \forall t > 0. \tag{2.1}$$

Definition 2.2. We say that the system (1.1) is asymptotically stable if any two solutions $y(t)$, $z(t)$ to (1.1) satisfy

$$\lim_{t \to +\infty} \|y(t) - z(t)\| = 0. \tag{2.2}$$

We note that 0 is an equilibrium if $f(t, 0, 0, 0) \equiv 0$. Then

(i) it follows from the contractivity of the system (1.1) that

$$\|y(t)\| \leq \max_{-\tau \leq s \leq 0} \|\phi(s)\|, \quad \forall t > 0, \tag{2.3}$$

which implies that the solutions to (1.1) are contractive;

(ii) it follows from the asymptotic stability of the system (1.1) that

$$\lim_{t \to +\infty} \|y(t)\| = 0, \tag{2.4}$$

which implies that the solutions to (1.1) are asymptotically stable and the system has only one equilibrium.

Now we introduce our definition of uniform ultimate boundedness. It can be found in [34,35] for non-neutral differential equations.