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A first order system least squares method for the Helmholtz equation

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ABSTRACT

We present a first order system least squares (FOSLS) method for the Helmholtz equation at high wave number k, which always leads to a Hermitian positive definite algebraic system. By utilizing a non-trivial solution decomposition to the dual FOSLS problem which is quite different from that of the standard finite element methods, we give an error analysis to the hp-version of the FOSLS method where the dependence on the mesh size h, the approximation order p, and the wave number k is given explicitly. In particular, under some assumption of the boundary of the domain, the L^2 norm error estimate of the scalar solution from the FOSLS method is shown to be quasi optimal under the condition that kh/p is sufficiently small and the polynomial degree p is at least $O(\log k)$. Numerical experiments are given to verify the theoretical results.

1. Introduction

Lots of least squares methods have been extensively studied for the efficient and accurate numerical approximation of many partial differential equations such as the elliptic, elasticity and Stokes equations. As mentioned in [1], there are three kinds of least-squares methods: the inverse approach, the div approach, and the div–curl approach. The interest of this paper is to consider the div approach least squares method which applies a chosen L^2 norm to a natural first order system for the Helmholtz equation with Robin boundary condition which is the first order approximation of the radiation condition:

$$-\Delta u - k^2 u = f \quad \text{in } \Omega, \tag{1.1a}$$

$$\frac{\partial u}{\partial \boldsymbol{n}} - \mathbf{i} k u = g \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) is a bounded, Lipschitz and connected domain, the wave number k is real and positive, and $\mathbf{i}$ denotes the imaginary unit. We want to point out that if the sign before $\mathbf{i}$ in (1.1b) is positive, the corresponding least squares method and theoretical analysis in this paper also hold. We impose further assumptions on the domain Ω in the following:

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(A1) There is a constant $C > 0$ such that for any $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$, the Helmholtz equation (1.1) has a unique solution $u \in H^1(\Omega)$ satisfying

$$\|\nabla u\|_{L^2(\Omega)} + k\|u\|_{L^2(\Omega)} \leq C\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right).$$

(A2) The boundary of Ω is analytic.

The above assumptions are intrinsic for the analysis in this paper, while the least squares method can be applied for more general cases. In fact, [2] shows the assumption (A1) holds if the domain Ω is star-shaped; and [3, Theorem 1.8] obtains the same estimate without the star-shaped restriction.

Due to the well-known pollution effect for the numerical solution of the Helmholtz equation, the standard Galerkin finite element methods can maintain a desired accuracy only if the mesh resolution is appropriately increased. Numerous nonstandard methods have been proposed in the literature to obtain more stable and accurate approximation, which includes quasi-stabilized finite element methods [4], absolutely stable discontinuous Galerkin (DG) methods [5–9], continuous interior penalty finite element methods [10,11], the partition of unity finite element methods [12,13], the ultra weak variational formulation [14], plane wave DG methods [15,16], spectral methods [17], generalized Galerkin/finite element methods [18–20], meshless methods [21], and the geometrical optics approach [22].

Generally, the linear systems from most of the above nonstandard Galerkin finite element approximations of the Helmholtz equation with high wave number k are strongly indefinite. But the least-squares Galerkin method for the Helmholtz equation always yields a Hermitian positive definite system [23,24]. Hence it attracts the design of an efficient solver. For instance, a div–curl approach least squares method was applied to the Helmholtz equation in [24], and an efficient solver based on wave-ray multigrid was proposed. Recently, numerical results in [25] show that a multiplicative Schwarz algorithm, without coarse solver, provides a p-preconditioner for solving the DPG system. The numerical observations suggest that the condition number of the preconditioned system is independent of the wavenumber k and the polynomial degree p. Since both DPG methods and FOSLS are residual minimization methods such that their linear systems are Hermitian positive definite, it is promising that the multiplicative Schwarz preconditioner in [25] will provide similar preconditioning for FOSLS. We will show the effect of the multiplicative Schwarz preconditioner for our FOSLS in a separate paper.

A key result revealed by J.M. Melenk and S. Sauter in [26] is that the polynomial degree p should be chosen in a wavenumber-dependent way to yield optimal convergent conditions. This important result was analyzed based on the standard Galerkin finite element method. It shows that, under the assumption that the solution operator for Helmholtz problems is polynomially bounded in k, quasi optimal convergence can be obtained under the conditions that kh/p is sufficiently small and the polynomial degree p is at least $O(\log k)$.

An objective of this paper is to extend the key result in [26] to the div approach FOSLS method, which will be called FOSLS method for brevity in the following. We use the standard Raviart–Thomas finite element space and continuous piece-wise polynomial finite element space for the discretization of the FOSLS method. The stability of the FOSLS solutions for the Helmholtz equation can be obtained by the property of FOSLS formulation and a Rellich-type identity approach. The main difficulty in the analysis lies in the establishment of quasi optimal convergence for the FOSLS method. We first mimic the technique proposed in [26] to decompose the Helmholtz solution into an oscillatory analytic part and a nonoscillatory elliptic part. A key estimate for the oscillatory analytic part of the Helmholtz solution (cf. (4.5c) in Theorem 4.3) is further derived for the error analysis of the FOSLS method for the Helmholtz equation. Another crucial estimate lies in the derivation of the dependence of convergence on the polynomial degree p. A new $H(\mathrm{div})$ projection is designed to overcome this problem, and some important estimates, which reveal the dependence of the projection error on k, h, p, for this $H(\mathrm{div})$ projection are obtained. In Remark 5.2, we explain why it is necessary to use Raviart–Thomas space instead of vector valued continuous piece-wise polynomial space to approximate vector fields in $H(\mathrm{div}, \Omega)$. In Remark 4.5, we give detailed explanation why the projection-based interpolation in [27] can *not* be applied for the quasi optimal convergent estimate for the Helmholtz equation. The most important part of the analysis lies in a modified duality argument for the FOSLS method which is motivated by the duality argument used in [1]. Roughly speaking, the corresponding dual FOSLS problem is to find $(\boldsymbol{\psi}, v) \in \{\boldsymbol{\psi} \in H(\mathrm{div}, \Omega) : \boldsymbol{\psi}\cdot\boldsymbol{n}|_{\partial\Omega} \in L^2(\partial\Omega)\} \times H^1(\Omega)$ satisfying

$$\begin{aligned}\|u-u_h\|^2_{L^2(\Omega)} &= (\mathbf{i}k(\boldsymbol{\phi}-\boldsymbol{\phi}_h) + \nabla(u-u_h), \mathbf{i}k\boldsymbol{\psi} + \nabla v)_\Omega + (\mathbf{i}k(u-u_h) + \nabla\cdot(\boldsymbol{\phi}-\boldsymbol{\phi}_h), \mathbf{i}kv + \nabla\cdot\boldsymbol{\psi})_\Omega \\ &\quad + k\langle(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\cdot\boldsymbol{n} + (u-u_h), \boldsymbol{\psi}\cdot\boldsymbol{n} + v\rangle_{\partial\Omega}.\end{aligned}$$

Here, $\mathbf{i}k\boldsymbol{\phi} + \nabla u = 0$, and $(\boldsymbol{\phi}_h, u_h)$ is the numerical approximation to $(\boldsymbol{\phi}, u)$. Then, the regularity estimates for the oscillatory analytic part $(\boldsymbol{\psi}_A, v_A)$ and the nonoscillatory elliptic part $(\boldsymbol{\psi}_{H^2}, v_{H^2})$ of the solution of the above dual FOSLS problem are deduced. Since the above dual FOSLS problem is quite different from the dual problem used in [28,26], these regularity estimates, especially the estimate of $\|\nabla\cdot\boldsymbol{\psi}_{H^2}\|_{H^1(\Omega)}$ (cf. (5.1e) in Lemma 5.1), gets involved with non-trivial modification to the original proof of solution decomposition in [26]. Finally the quasi optimality of the L^2 norm error estimate for the scalar solution of the FOSLS method for the Helmholtz equation can be finally obtained under the conditions that kh/p is sufficiently small and the polynomial degree p is at least $O(\log k)$.

We want to emphasize that FOSLS is closely related to the discontinuous Petrov–Galerkin (DPG) methods, see [29–45]. Recently, the DPG$_\varepsilon$ method, which is of the least-squares type, was proposed in [46]. The DPG$_\varepsilon$ solution may yield less pollution error than the general FOSLS with fixed polynomial degree p and on the same mesh. The analysis for FOSLS in this