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VaR as the CVaR sensitivity: Applications in risk optimization

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ABSTRACT

VaR minimization is a complex problem playing a critical role in many actuarial and financial applications of mathematical programming. The usual methods of convex programming do not apply due to the lack of sub-additivity. The usual methods of differentiable programming do not apply either, due to the lack of continuity. Taking into account that the CVaR may be given as an integral of VaR, one has that VaR becomes a first order mathematical derivative of CVaR. This property will enable us to give accurate approximations in VaR optimization, since the optimization VaR and CVaR will become quite closely related topics. Applications in both finance and insurance will be given.

1. Introduction

VaR has many applications in finance and insurance. Risk management, capital requirements, financial reporting, asset allocation, *bonus-malus* systems, optimal reinsurance, etc. just compose a brief list of topics closely related to *VaR*. Beyond *VaR*, risk measurement is an open problem provoking a growing interest and discussion in recent years. Since Artzner et al. [1] introduced their coherent measures of risk, much more approaches have been proposed. Very important examples are the expectation bounded measures of risk [2], consistent risk measures [3], actuarial risk measures [4], indices of riskiness [5–7], etc.

The existence of alternative risk measures implies that many risk-linked problems may be studied without dealing with *VaR*. Moreover, *VaR* is not sub-additive [1], it is difficult to optimize [8] and it presents some more drawbacks which may recommend to deal with other risk measures such as *CVaR* [9]. Nevertheless, for several reasons *VaR* still plays a

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critical role for many practitioners, institutions and researchers. Firstly, regulation (Basel for banks, Solvency for insurers, etc.) still assigns a vital role to *VaR*. Secondly, *VaR* is never infinity, while the rest of usual risk measures may attain this value. For instance, *CVaR* becomes infinite for random risks whose expected losses equal infinity (for instance, positive random variables with unbounded expectation). Infinite values may provoke analytical and mathematical problems quite difficult to overcome, specially if several heavy tails are simultaneously involved [10]. Heavy tails are usual in some actuarial topics [11], some operational risk topics [12] and other issues. Thirdly, sub-additivity may be undesirable for some actuarial and financial problems, as pointed out by Dhaene et al. [13], who suggested the use of *VaR* for some merger-linked problems, for instance. Fourthly, for very important financial problems, *VaR* often provides valuable solutions from both theoretical [14] and empirical [15] viewpoints, and *VaR* also facilitates the use of probabilities in both the objective function and/or the constraints of several financial optimization problems (Dupacová and Kopa [16], Zhao and Xiao [17], etc.).

The optimization of *VaR* is much more complicated than the optimization of other risk measures (Rockafellar and Uryasev [9], Larsen et al. [18], Gaivoronski and Pflug [8], Shaw [19], Wozabal [20], etc.). Since *VaR* is neither convex nor differentiable, one may face the existence of many local minima, and they may become undetectable by means of the standard optimization methods. There are many and quite different approaches addressing the optimization of *VaR* (Larsen et al. [18], Gaivoronski and Pflug [8], Shaw [19], Wozabal [20], etc.). All of them yield interesting algorithms or optimality conditions allowing us to find adequate solutions under different assumptions, but none of them solves the problem in an exhaustive manner. There are many cases which cannot be treated with the existent methodologies.

A very interesting approach may be found in [21,20]. The authors deal with discrete probability spaces composed of finitely many atoms, and they prove that *VaR* equals the difference of two convex functions. This property allows them to provide efficient optimizing algorithms. Nevertheless, it is easy to show that the property above does not hold for general probability spaces. Since there are many problems involving *VaR* and continuous random variables (Shaw [19], Zhao and Xiao [17], etc.), further extensions containing general probability spaces should be welcome.

This paper deals with a very simple idea. If the *CVaR* (also called *AVaR*, or average value at risk) may be given as an integral of *VaR*, then *VaR* must become a first order mathematical derivative of *CVaR*. Consequently, an approximation of *VaR* must be given by the change in *CVaR* over the change in level of confidence (or, in other words, by a quotient of increments). Hence, an approximation of *VaR* must be given by the difference of two convex functionals, and the result of Wozabal [20] will become true in general probability spaces if one takes a limit.

Ideas above will be formalized in Section 2, where it will be proved that *VaR* is the limit of the difference of convex functionals. We will also explain why one does not need to take any limit in the discrete case. In Section 3 we will consider a sequence of optimization problems whose objective function has a limit, and we will analyze the relationship between the sequence of solutions and the solution optimizing the limit. As a consequence, we will establish conditions under which the optimization of *VaR* may be solved by optimizing the difference of two convex functionals. In Section 4 we will focus on a methodology proposed in [22] and we will address the minimization of the difference of two convex functionals in arbitrary probability spaces. Several optimality conditions will be found. Applications in finance (optimal investment) and insurance (optimal reinsurance) will be given in Section 5. Though the purpose of Section 5 is merely illustrative, these examples will be general enough, since they will apply in both static and dynamic frameworks and for discrete or continuous price/claim processes. Section 6 will summarize the paper.

2. Preliminaries and notations

We will deal with the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ composed of the set Ω, the σ-algebra $\mathcal{F}$ and the probability measure $\mathbb{P}$. We can consider $1 \leq p < \infty$ and the space L^p (also denoted by $L^p(\mathbb{P})$ or $L^p(\Omega, \mathcal{F}, \mathbb{P})$) of real-valued random variables y such that $\mathbb{E}\left(|y|^p\right) < \infty$, $\mathbb{E}()$ representing the mathematical expectation. Recall that L^q is the dual space of L^p, where $1 < q \leq \infty$, $1/p + 1/q = 1$, and L^∞ is composed of the essentially bounded random variables (Riesz Representation Theorem, Rudin [23]). Recall also that the usual norm of L^p is

$$\|y\|_p := \left(\mathbb{E}\left(|y|^p\right)\right)^{1/p} \tag{1}$$

if $1 \leq p < \infty$ and $\|y\|_\infty :=$ Ess_Sup $(|y|)$, Ess_Sup denoting "essential supremum".

For $1 \leq p \leq p' \leq \infty$ we have that $L^p \supset L^{p'}$. In particular, $L^1 \supset L^p \supset L^\infty$ for every $1 \leq p \leq \infty$. Recall also that for $1 \leq p \leq \infty$ we have that L^p may be endowed with the topology $\sigma(L^p, L^q)$, which is weaker than the norm topology. Furthermore, if $1 < p < \infty$ then every convex, closed and bounded subset of L^p is $\sigma(L^p, L^q)$-compact (Hahn–Banach's Theorem and Alaoglu's Theorem). If Ω is a finite set then L^p becomes a finite-dimensional space for every $1 \leq p \leq \infty$, $L^p = L^{p'}$ for every $1 \leq p \leq p' \leq \infty$, and all of the introduced topologies of L^p coincide. Further details about Banach spaces of random variables may be found in [24,23,25].

The space L^0 containing every real-valued random variable may be endowed with the usual convergence in probability, in which case L^0 becomes a metric (but not Banach) space. The usual distance in L^0 is given by $d(y, z) = \mathbb{E}(\mathrm{Min}(1, |y - z|))$, and it is known that $L^1 \subset L^0$ [23].

Finally, we will deal with many topological properties. All of them may be found in [26].