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On the generalized cumulative residual entropy with applications in actuarial science

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ABSTRACT

Recently, Psarrakos and Navarro (2013) proposed a new measure of uncertainty which extends the cumulative residual entropy (CRE), called the generalized cumulative residual entropy (GCRE). In the present paper, new properties and applications in actuarial risk measures of the GCRE are explored. Bounds, stochastic order properties and characterization results of the new entropy are also discussed. It is shown that the GCRE is invariant under changes in location, and scale directly with scale of a random variable; the same properties also hold for the standard deviation. It is also proved that the GCRE of the first order statistics can uniquely determine the parent distribution. The Weibull distribution, which is commonly used in several fields of applied probability, is characterized by using the mentioned generalized measure. The GCRE is studied as a risk measure and is compared to the standard deviation and the right-tail risk measure, where the latter measure was introduced by Wang (1998). Several examples are also given to illustrate the new results.

1. Introduction

In the last decades, there has been a great deal of interest in the measurement of uncertainty associated to a probability distribution. In information theory, the concept of entropy is of fundamental importance measure and it was first introduced by Shannon [1]. Generally, entropy function is a useful tool in actuarial science, see for example [2–4]. Furthermore, several authors have studied the concept of entropy as a dispersion measure, see e.g. [5,6] for a greater detail. This paper deals with the generalized cumulative residual entropy (see [7]) and gives some applications in actuarial science. Note that the cumulative residual entropy was first introduced by Rao et al. [8] and recently was studied as a risk measure by Yang [9].

Let X be an absolutely continuous nonnegative random variable with probability density function (pdf) f and cumulative distribution function (cdf) F. The Shannon entropy of X is defined as

$$H(X) = -E(\log f(X)) = -\int_0^\infty f(x)\ \log f(x)\, dx, \tag{1}$$

where "log" is the natural logarithm with the convention $0 \log 0 = 0$. For the basic properties and applications of Shannon entropy, we refer the reader to Cover and Thomas [10]. For a nonnegative continuous random variable X, Rao et al. [8] (see

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also [11]) proposed an alternate information measure, replacing the probability density function in Shannon entropy with the survival function, called the cumulative residual entropy (CRE). This entropy is defined as

$$\mathcal{E}(X) = -\int_0^\infty \bar{F}(x)\log\bar{F}(x)\,dx = \int_0^\infty \bar{F}(x)\Lambda(x)\,dx, \tag{2}$$

where $\bar{F}(x) = 1 - F(x)$ is the survival (reliability) function and

$$\Lambda(x) = -\log\bar{F}(x), \quad x \geq 0,$$

is the cumulative hazard function. Recently, Psarrakos and Navarro [7] extended the CRE measure by the generalized cumulative residual entropy (GCRE) of X, defined as

$$\mathcal{E}_n(X) = \frac{1}{n!}\int_0^\infty \bar{F}(x)[\Lambda(x)]^n\,dx, \tag{3}$$

for all $n = 0, 1, 2\ldots$ Note that for $n = 0$, we obtain the mean value $\mathcal{E}_0(X) = E(X)$, while for $n = 1$, we get $\mathcal{E}_1(X) = \mathcal{E}(X)$. Some examples of GCRE are given in the following example.

Example 1.1. (a) If X is uniformly distributed in $[0, b]$, $b > 0$, then it is easy to see that $\mathcal{E}_n(X) = b\,n!/2^{n+1}$ for all integers $n \geq 0$.

(b) If X has a Weibull distribution with survival function $\bar{F}(x) = e^{-\lambda x^\gamma}$, $x \geq 0$, $\lambda, \gamma > 0$, for all integers $n \geq 0$, then we obtain

$$\mathcal{E}_n(X) = \frac{\Gamma\left(n + \frac{1}{\gamma}\right)}{n!\gamma\lambda^{\frac{1}{\gamma}}},$$

where $\Gamma(\cdot)$ is the complete gamma function.

(c) If X has a Pareto (Lomax) distribution with survival function $\bar{F}(x) = (\frac{\beta}{x+\beta})^\alpha, x \geq 0, \alpha > 1, \beta > 0$, then $\mathcal{E}_n(X) = \beta\alpha^n/(\alpha-1)^{n+1}$, for all integers $n \geq 0$.

We recall that a random variable X is said to be smaller than a random variable Y in the usual stochastic order, denoted by $X \leq_{st} Y$, if $E[\Phi(X)] \leq E[\Phi(Y)]$ for all increasing functions Φ such that the expectations exist. A random variable X is said to be smaller than a random variable Y in the increasing convex order, denoted by $X \leq_{icx} Y$, if $E[\Phi(X)] \leq E[\Phi(Y)]$ for all increasing convex functions Φ such that the expectations exist. Moreover, if $\overline{F}(x)$ and $\overline{G}(x)$ are the survival functions of X and Y, respectively, then we say that X is smaller than Y in the hazard rate order, denoted by $X \leq_{hr} Y$, if $\overline{G}(x)/\overline{F}(x)$ is increasing with respect to x. Notice that throughout this paper, the terms 'increasing' and 'decreasing' are used in a non-strict sense. The connection between the earlier mentioned stochastic orders is described in the following diagram

$$X \leq_{hr} Y \quad\Rightarrow\quad X \leq_{st} Y \quad\Rightarrow\quad X \leq_{icx} Y.$$

Further properties and applications can be found in the book of Shaked and Shanthikumar [12].

The concept of GCRE is related to the mean time between the (upper) record values of a sequence of independent and identically distributed (i.i.d.) nonnegative random variables and with the concept of relevation transform. In [7] (see also [13]), basic properties of GCRE and relations with other types of functions and entropies are presented. They also studied the GCRE of the random variable $X - t \mid X > t$ as a function of $t \geq 0$, called the dynamic GCRE (see relation (14)). For a fixed $n \in \{1, 2, \ldots\}$ and all t, they also derived characterization results for absolutely continuous distributions.

The aim of the present paper is two-fold: First, we provide some new distributional properties of GCRE, and then we apply it as a risk measure. In particular, we derive various bounds, alternative expressions of GCRE, stochastic ordering properties and characterization results. Moreover, the GCRE of the proportional hazard rates model was discussed which arises naturally in different fields of applied probability. Analogous properties have been obtained by Di Crescenzo and Longobardi [14] for the cumulative residual past entropy. We also study the GCRE as a risk measure since it is shift-independent while it is scale dependent (see Proposition 2.1). It is well-known that the latter two properties also hold for the standard deviation, which usually arises in real-life problems. Note that in the case of exponential distribution, GCRE and standard deviation are the same.

As pointed out by Wang [15], see also [16], there is a poor performance of standard deviation for measuring large insurance risk with long tailed skewed distributions. This motivated us to investigate the performance of the GCRE for long tailed distributions, such as Pareto, Exponential-Inverse Gaussian, Weibull with shape parameter $\gamma \in (0, 1)$. In order to examine the contribution of GCRE measure, we compare the GCRE with the right-tail risk measure, introduced by Wang [15]. Also, some other known measures in actuarial sciences which affect by the right tail of a distribution in the real-life problem, were applied in the sequel. Generally, the GCRE can be used as a useful tool in real-life applications of contemporary numerical techniques in the different fields, see e.g., [17–22], and the references therein.

Therefore, the rest of this paper is organized as follows: In Section 2, we discuss some new basic properties of GCRE such as the effect of linear transformation, and stochastic order properties. We also construct bounds for GCRE and discuss the