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Deterministic impulse control problems: Two discrete approximations of the quasi-variational inequality

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A B S T R A C T

In this paper, we study a deterministic infinite horizon, mixed continuous and impulse control problem in $\mathbb{R}^n$, with general impulses, and cost of impulses. We assume that the cost of impulses is a positive function. We prove that the value function of the control problem is the unique viscosity solution of the related first order Hamilton–Jacobi quasi-variational inequality.[1] We then propose time discretization schemes of this QVI, where we consider two approximations of the "Hamiltonian hH", including a natural one. We prove that the approximate value function u^h exists, that it is the unique solution of the approximate QVI and that it forms a uniformly bounded and uniformly equicontinuous family. We also prove that the approximate value function converges locally uniformly, towards the value function of the control problem, when the discretization step h goes to zero; the rate of convergence is proved to be in h^σ, where $0 < \sigma < 1/2$.

1. Introduction

This paper extends to the quasi variational inequality of impulse control problems earlier works on the discretization of Hamilton–Jacobi equations [1,2]. We treat in the paper a mixed continuous and impulse control problem, with general impulses and cost of impulses, and we will refer to it as an impulse control problem. We focus on an infinite horizon impulse control problem, but our approach being a time discretization with a Lagrangian differentiation, it may, with minimal adaptation, be extended to finite horizon impulse control problems. We give regularity results for the value function of the impulse control problem considered, and show that it is the unique bounded and uniformly continuous[2] viscosity solution of the associated QVI. We propose two related discretization schemes, and for both show the convergence of the approximate value function to the viscosity solution of the QVI, i.e. the value function of the impulse control problem, and prove a rate of convergence in a positive (although lower than one) power of the step size. This convergence rate also depends on the regularity of the value function, Lipschitz or Hölder continuous, depending on the data.

In the previous literature, authors in [3] use the approach of changing the original infinite horizon impulsive control problem into another one equivalent without impulses, by adding one state variable, in order to use the classic dynamic programming theory. In this paper, they consider a "slow growth" problem. The meaning of the existence and uniqueness of the solution of the differential equation considered is not, however, clearly detailed.

In the case of finite horizon impulsive control [4], the discretization is based on Euler scheme with step h. The convergence rate is in $h^{\frac{1}{2}}$.

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1 QVI.

2 BUC.

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In [5,6], where weaker assumptions on data are examined, the authors consider some general impulse control problems with a stopping time control, in a set $Q = \Omega \times [0, T]$, where $\Omega \subset \mathbb{R}^n$ is an open bounded set. They assume that the number of impulses is finite and that both continuous and impulse controls have values in compact sets. Based on the fact that the value function is characterized as the maximum element of a set W, their procedure consists in approximating the set Q with a triangulation Q^h, in considering a set W^h of linear finite elements defined from Q^h to $\mathbb{R}$ and satisfying some conditions on the vertices and on sets approximating the admissible controls, and then in finding the maximum element $\bar{w}^h$ of this set W^h, with respect to a certain order "$\leq$". The authors prove that the approximate value function $\bar{w}^h(x,t)$ converges uniformly to the value function $V(x,t)$. In [7], the authors study the same problem in the stationary case. The approximation procedure, where the set Ω is approximated with a triangulation Ω^h, is the same than above. They prove that the approximate value function $\bar{w}^h(x)$ converges uniformly to the value function $V(x)$. Ignoring the continuous control in their control policy, they prove that the convergence rate is in $|\log \|h\|| \, \|h\|^{\frac{1}{2}}$, for all $\|h\| \leq \|h_0\|$, h_0 being a fixed "initial" triangulation.

We also cite [8] where the author deals with explicit and implicit finite difference schemes to the viscosity solution of Hamilton–Jacobi equations in finite horizon.

The paper is organized as follows: we first present the general impulse control problem studied, the associated first order Hamilton–Jacobi QVI, the assumptions on the data, and give results on the regularity of the value function. We then prove that the value function is the unique BUC viscosity solution of the related QVI. In Section 5, we expose the discrete approximations of the QVI, and prove that the approximate value function u^h is their unique solution, that it forms a uniformly bounded and uniformly equicontinuous family. In Section 6, we prove that the approximate value function converges, locally uniformly towards the value function, as the step h goes to zero and establish the convergence rate.

2. Statement of the problem

We consider a deterministic, mixed continuous and impulse control problem in $\mathbb{R}^n$, and general in term of the form of impulses, and cost of impulses

$$\begin{cases} \dot{y}_x(t) &= b(y_x(t), \tau(t)), \quad t \neq t_k \\ y_x(0) &= x, \\ y_x(t_k^+) &= g(y_x(t_k^-), \xi_k), \end{cases}$$

where b is a function from $\mathbb{R}^n \times \mathbb{R}^l$ into $\mathbb{R}^n$, $\tau(t)$ being the continuous control, is any measurable function from $\mathbb{R}^+$ to $\mathbb{R}^l$, $(\{t_k\}, \{\xi_k\})$ being the impulse control, where $(t_k)_{k\in\mathbb{N}}$ is a nondecreasing sequence of nonnegative real numbers which satisfies: $t_k \to +\infty$ when $k \to +\infty$, $(\xi_k)_{k\in\mathbb{N}}$ is a sequence of elements of $\mathbb{R}^m$, and g is a function from $\mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^n$.

Let $\Psi = (\tau(.), \{t_k\}, \{\xi_k\})$ be the mixed continuous and impulse control. The payoff J is defined as

$$J(x, \Psi) = \int_0^\infty f(y_x(t), \tau(t)) \exp(-\lambda t) dt + \sum_{k\in\mathbb{N}} c(y_x(t_k^-), \xi_k) \exp(-\lambda t_k),$$

where λ is a positive number. Let $u(x) = \inf_\Psi J(x, \Psi)$, be the value function of this problem.

The classic QVI associated to this impulse control problem is the following

$$\max\{H(x, u, Du), u - Mu\} = 0 \quad \text{in } \mathbb{R}^n, \tag{1}$$

where

$$H(x, u, Du) = \sup_{\tau\in\mathbb{R}^l} (-b(x, \tau).Du + \lambda u - f(x, \tau)),$$

$$Mu(x) = \inf_{\xi\in\mathbb{R}^m} (u(g(x, \xi)) + c(x, \xi)).$$

We will consider in the paper the space $\mathcal{X}$ of bounded and continuous functions on $\mathbb{R}^n$, normed by $\|v\|_\infty = \sup_{x\in\mathbb{R}^n} |v(x)|$.

Remark 2.1. The typical example that provides an interesting framework of the theory of infinite horizon impulse control is the inventory control problem or the management of stocks [9]. We give hereafter an illustrative example:

$$\begin{cases} \dot{y}_x(t) &= -\tau(t), \quad t \in]t_k, t_{k+1}[, \\ y_x(0) &= x, \\ y_x(t_k^+) &= y_x(t_k^-) + \xi_k, \end{cases}$$

x denotes the initial inventory, $\tau(t)$ is a measurable function from $\mathbb{R}^+$ to a compact set $\mathcal{K} \subset (\mathbb{R}^+)^n$ that corresponds to the instantaneous demand of the goods, the times t_k are the times where it is decided to replenish the inventory and $\xi_k \in (\mathbb{R}^+)^n$ are the levels of replenishment.

We propose the following payoff J where the cost of storage and out of stocks is of integral type, we allow that the cost f depends also on the demand $\tau(t)$. The typical example is the cost of handling the goods.

$$J(x, \Psi) = \int_0^\infty f(y_x(t), \tau(t)) \exp(-\lambda t) dt + \sum_{k\in\mathbb{N}} c(y_x(t_k^-), \xi_k) \exp(-\lambda t_k).$$