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Preserving monotony of combined edge finite volume–finite element scheme for a bone healing model on general mesh

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ABSTRACT

In this article, we propose and analyze a combined finite volume–finite element scheme for a bone healing model. This choice of discretization allows to take into account anisotropic diffusions without imposing any restrictions on the mesh. Moreover, following the work of Cancès et al. (2013), we define a nonlinear correction of the diffusive terms to obtain a monotone scheme. We provide, under a numerical assumption, a complete convergence analysis of this corrected scheme, and present some numerical experiments which show its good behavior.

1. Introduction

We consider a model taking into account the main biological phenomena acting in bone healing. It describes the evolution of the concentrations of the following four quantities: the mesenchymal stem cells (denoted s), the osteoblasts (denoted b), the bone matrix (denoted m) and the osteogenic growth factor (denoted g). Bone healing begins by the migration of the stem cells to the site of the injury. Then along the bone, these cells differentiate into osteoblasts which start to synthetize the bone matrix. This cell differentiation is only possible in presence of the growth factor.

The proposed model is based on that described in [1]. It takes into account several phenomena: the diffusion of the stem cells and the growth factor, the migration of the stem cells towards the bone matrix, the proliferation and the differentiation of the stem cells. The osteoblasts are considered without movement since they are fixed at the bone matrix. Moreover, the model includes the case of heterogeneous domains, with possibly anisotropic diffusions.

In this paper, we propose and analyze a numerical scheme for this bone growth model. This scheme was already introduced in [2], but without any convergence study. A finite volume scheme was previously proposed in [3] for this model in homogeneous domains where the diffusion tensor is considered to be the identity matrix. Moreover, the convergence analysis is performed only for meshes assumed to be admissible [4, Definition 9.1].

On the one hand, the classical cell-centered finite volume method with an upwind discretization of the convective terms provides the stability and is extremely robust. In this case, the mesh is assumed to be admissible. In particular, it implies that the orthogonality condition has to be satisfied. As mentioned in [3], a difficulty in the implementation is to construct such admissible meshes. Structured rectangular meshes are admissible, but they cannot be used for complex geometries arising in physical contexts. However, standard finite volume schemes do not permit to handle anisotropic diffusion on general

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meshes due to the nonconsistency of the numerical flux. A large variety of methods have been proposed to reconstruct a consistent gradient, see for example [5] and references therein.

On the other hand, the finite element method allows for an easy discretization of diffusive terms with full tensors without imposing any restrictions on the meshes. It was used a lot for the discretization of degenerate parabolic equations. For example, conforming piecewise linear finite element method has been studied in [6], as well as mixed finite element method in [7]. However, some numerical instabilities may arise in the convection dominated case.

The idea is hence to combine a finite element discretization of diffusive terms with a finite volume discretization of the other terms. Such schemes were proposed and studied in [8] for fluid mechanics equations where the diffusion tensor is the identity matrix. This method was then extended in [9] to inhomogeneous and anisotropic diffusion–dispersion tensors and to very general meshes only satisfying the shape regularity condition (6). Such discretizations were then applied to different physical models, such as the anisotropic Keller–Segel chemotaxis system [10] or the two compressible phase flow in porous media [11]. However, it is well-known that the discrete maximum principle is no more guaranteed if there exist negative transmissibilities. A nonlinear stabilization term is introduced in [12] to design a Galerkin approximation of the Laplacian, but heterogeneous anisotropic tensors are not considered. More recently, a general approach to construct a nonlinear correction providing a discrete maximum principle was proposed in [13,14]. This method allows to maintain some crucial properties of the initial scheme, in particular coercivity and convergence towards the weak solution of the continuous problem as the size of the mesh tends to zero. We will apply it to the diffusive terms of our combined scheme to ensure a discrete maximum principle.

The outline of this article is as follows. In Section 2, we introduce the considered bone healing model. In Section 3, we define the combined finite volume–finite element scheme and we apply the method described in [13] to construct a nonlinear correction providing a discrete maximum principle. Then in Section 4 we prove the existence of a physically admissible solution and give some discrete a priori estimates. Thanks to these estimates, we prove in Section 5 the compactness of a family of approximate solutions. It yields the convergence (up to a subsequence) of the solution of the scheme to a solution of the continuous system as the size of the discretization tends to zero. Finally in Section 6 we present some numerical experiments showing the efficiency of the scheme.

2. The bone healing model

We consider the following model for bone healing: for $t \in (0, T)$ and $x \in \Omega$, where $T > 0$ and Ω is an open bounded domain of $\mathbb{R}^d$ ($d = 2,\ 3$),

$$\partial_t s - \operatorname{div}\left[\mathbf{D}(x)\left(\Lambda(m)\nabla s - V(m)\chi(s)\nabla m\right)\right] = K_1(m)\chi(s) - H(g)s =: f_1(s, m, g), \tag{1}$$

$$\partial_t b = K_2(m)\chi(b) + \rho H(g)s - \delta_1 b =: f_2(s, b, m, g), \tag{2}$$

$$\partial_t m = \lambda(1 - m)b =: f_3(b, m), \tag{3}$$

$$\partial_t g - \operatorname{div}\left(\mathbf{D}(x)\Lambda_g \nabla g\right) = P(g)b - \delta_2 g =: f_4(b, g). \tag{4}$$

The diffusion coefficient $\Lambda(m)$ and haptotaxis velocity $V(m)$ for the stem cells are given by

$$\Lambda(m) = \frac{\chi_h}{\zeta_h^2 + m^2}(m + \lambda_0)(1 - m + \lambda_0), \qquad V(m) = \frac{\chi_k}{(\zeta_k + m)^2}.$$

The factors $K_i(m)$, $i = 1,\ 2$ taking part in the mitosis terms are defined by

$$K_i(m) = \frac{\alpha_i}{\beta_i^2 + m^2}\, m.$$

The accumulation of stem cells and osteoblasts is limited by the multiplicative term $\chi(s) = s(1 - s)$. Moreover, the differentiation coefficient is given by

$$H(g) = \frac{\gamma_1}{\eta_1 + g} g,$$

and the production term of the growth factor is defined as

$$P(g) = \frac{\gamma_2}{(\eta_2 + g)^2} g.$$

The parameters $\alpha_i,\ \beta_i,\ \gamma_i,\ \eta_i,\ \delta_i\,(i = 1,\ 2)$, $\rho, \lambda, \chi_h,\ \zeta_h,\ \lambda_0,\ \chi_k,\ \zeta_k,\ \Lambda_g$ are given positive numbers. Finally, we assume that the permeability $\mathbf{D} : \Omega \to \mathcal{M}_d(\mathbb{R})$, where $\mathcal{M}_d(\mathbb{R})$ is the set of symmetric matrices $d \times d$, verifies:

$$D_{i,j} \in L^\infty(\Omega) \quad \forall i,\ j = 1, \ldots, d,$$

and that there exists $C_D > 0$ such that a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^d$,

$$\mathbf{D}(x)\xi \cdot \xi \geq C_D|\xi|^2.$$

This diffusion tensor $\mathbf{D}$ allows to take into account the heterogeneity of the biological domain.