Journal of Computational and Applied Mathematics 309 (2017) 383–395

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Mean square solution of Bessel differential equation with uncertainties

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ARTICLE INFO

Article history:
Received 21 September 2015
Received in revised form 10 January 2016

Keywords:
Random differential equation
L_p-random calculus
Bessel differential equation

ABSTRACT

This paper deals with the study of a Bessel-type differential equation where input parameters (coefficient and initial conditions) are assumed to be random variables. Using the so-called L_p-random calculus and assuming moment conditions on the random variables in the equation, a mean square convergent generalized power series solution is constructed. As a result of this convergence, the sequences of the mean and standard deviation obtained from the truncated power series solution are convergent as well. The results obtained in the random framework extend their deterministic counterpart. The theory is illustrated in two examples in which several distributions on the random inputs are assumed. Finally, we show through examples that the proposed method is computationally faster than Monte Carlo method.

1. Introduction

Deterministic differential equations have demonstrated to be powerful tools to model a number of problems in physics, chemistry, epidemiology, engineering, etc. When they are put in practice, their inputs (coefficients, forcing term, initial/boundary conditions) need to be set from sampled data, which usually contain uncertainty. The main source of randomness comes from measurement errors and complexity of the phenomenon under analysis. This leads to two main approaches in dealing with differential equations with randomness, namely, stochastic differential equations and random differential equations. On the one hand, stochastic differential equations consider uncertainty through an irregular Gaussian stochastic process termed as white noise, i.e., the derivative of the Wiener process. Their analytic and numerical study requires the so-called Itô calculus [1,2]. On the other hand, random differential equations constitute natural extensions of their deterministic counterpart since the involved input parameters are considered directly random variables and/or stochastic process having a more regular behavior. The advantage of considering random differential equations against stochastic differential equations is the wide range of well-known probability distributions that can be assigned to their input parameters such as beta, gamma, lognormal and Gaussian [3–7]. The analysis of random differential equations is based on the so-called L_p-random calculus, being mean square and mean fourth calculus specializations corresponding to $p = 2$ and $p = 4$, respectively, that have demonstrated to be very useful for this purpose [8,9].

The goal of this paper is to construct a mean square solution for the Bessel random differential equation (r.d.e.)

$$t^2\ddot{X}(t) + t\dot{X}(t) + (t^2 - A^2)X(t) = 0, \quad t > 0, \tag{1}$$

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http://dx.doi.org/10.1016/j.cam.2016.01.034
0377-0427/

where A is assumed to be a random variable defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Throughout the paper, we will assume that A is a non-negative random variable with probability 1 (w.p. 1), i.e.,

$$\mathbb{P}\left[\{\omega \in \Omega : A(\omega) \geq 0\}\right] = 1. \tag{2}$$

The construction of such solution will be performed by random generalized power series whose mean square convergence will be justified taking advantage of L_p-random calculus. From an applied point of view, it is important to point out that the computation of the rigorous solution of (1) in the mean square sense guarantees that the approximations generated by truncating the exact random power series solution of (1) will converge to the corresponding exact mean and variance. These two statistical moments are often the most relevant information required in applications. This advantage makes L_p-random calculus, and hence mean square convergence, the convenient framework to study random differential equation (1) instead of using alternative stochastic convergences such as almost surely, in probability and distribution. Furthermore we shall show later, through several numerical examples, that random generalized power series solution approach is faster than Monte Carlo sampling. This latter approach is the most widely used method to deal with random differential equations in applications.

The consideration of randomness in the A parameter that appears in the Bessel differential equation (1) can be motivated from physical considerations. The wave propagation generated by a electric field and its variations in the medium can be considered as being randomly varying due to inhomogeneous physical properties of the medium. As it is shown in [10], the governing equation for the electric field in a specific direction is given by a Bessel equation of the form (1), where A coefficient depends upon random medium parameters. From a mathematical point of view the Bessel differential equation is encountered when solving boundary value problems, such as separable solutions to Helmholtz equation in cylindrical or spherical coordinates. The A parameter determines the order of the Bessel functions found in the solution of Eq. (1). In the deterministic framework A parameter can take any real value. A natural generalization of this equation to the random context consists of assuming that A parameter together with the corresponding initial conditions are random variables rather than deterministic numbers. The extension to the random scenario of another classical second-order linear differential equations that appear in physics can be found in [11] and in the references therein. In [11], the study is conducted taking advantage of L_p-calculus. Another contributions solving random differential equations in the mean square sense include [12–14].

The paper is organized as follows. In Section 2 the main results regarding the so-called L_p-random calculus that will be required throughout the paper are summarized and/or established. Section 3 is devoted to construct two mean square convergent random generalized power series of the Bessel differential equation under mild conditions. Section 4 is addressed to apply the theoretical results established in Section 3 to construct a mean square solution of the random Bessel differential equation with two random initial conditions. Several illustrative examples are shown in Section 5. Conclusions are drawn in Section 6.

2. Preliminaries on L_p-random calculus

Hereinafter, the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a complete probability space. For the sake of clarity, first we will summarize the main definitions and results that will be used throughout this paper. Further details about them can be found in [1,8,9,15]. We will also establish new technical results related to the so-called L_p-random calculus that will be required later.

Let $p \geq 1$ be a real number. A real random variable X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called of order p (in short, p-r.v.), if

$$\mathrm{E}\left[|X|^p\right] < \infty,$$

where $\mathrm{E}[\ ]$ denotes the expectation operator. The set $\mathrm{L}_p(\Omega)$ of all the p-r.v.'s endowed with the norm

$$\|X\|_p = \left(\mathrm{E}\left[|X|^p\right]\right)^{1/p},$$

is a Banach space, [16, p. 9]. Let $\{X_n : n \geq 0\}$ be a sequence in $\mathrm{L}_p(\Omega)$. We say that it is convergent in the pth mean to $X \in \mathrm{L}_p(\Omega)$, if

$$\lim_{n\to\infty} \|X_n - X\|_p = 0.$$

This convergence is denoted by $X_n \xrightarrow[n\to+\infty]{p\text{th mean}} X$. For $p = 2$, this 2th mean convergence is usually referred to as mean square convergence.

If $q > p \geq 1$, and $\{X_n : n \geq 0\}$ is a convergent sequence in $\mathrm{L}_q(\Omega)$, that is, qth mean convergent to $X \in \mathrm{L}_q(\Omega)$, then $\{X_n : n \geq 0\}$ is in $\mathrm{L}_p(\Omega)$ and it is pth mean convergent to $X \in \mathrm{L}_p(\Omega)$. In general, $\mathrm{L}_q(\Omega) \subset \mathrm{L}_p(\Omega)$ for $q > p \geq 1$, [16, p. 13]. Moreover, using the Cauchy–Schwarz inequality one can demonstrate that [17, p. 415]

$$\|XY\|_q \leq \|X\|_{2q}\|Y\|_{2q}, \quad X, Y \in L_{2q}(\Omega),\ q \geq 1. \tag{3}$$

From these facts it is easy to establish the following.

Proposition 1. *Let $\{X_n : n \geq 0\}$ be a sequence in $\mathrm{L}_{2q}(\Omega)$, $q \geq 1$. If $Y \in \mathrm{L}_{2q}(\Omega)$ and $X_n \xrightarrow[n\to+\infty]{2q\text{th mean}} X$ then, $YX_n \xrightarrow[n\to+\infty]{q\text{th mean}} YX$.*